BRINGING OUT THE ALGEBRAIC CHARACTER OF ARITHMETIC

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*First Draft of a paper presented at symposium, “Toward a research base for algebra reform beginning in the early grades” during the Annual Meeting of the American Educational Research Association, Montreal, Canada, April 21, 1999. Partial support was provided by NSF Grant #9722732, "Bridging Arithmetic and Algebra", awarded to Analúcia D. Schliemann. Support for the multimedia version of this paper was provided by NSF-Grant 9805289, "Bridging Research and Practice", awarded to Ricardo Nemirovsky & David Carraher. Special thanks go to Thelma Davis and John F. O'Meara of East Somerville Community School, Somervile, MA and to Cara DiMattia of TERC.

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EARLY ALGEBRA

For many years, educational research about algebra highlighted the considerable difficulties adolescents display in understanding and using algebra (Booth, 1984; Da Rocha Falcão, 1992; Filloy & Rojano, 1989; Kieran, 1985a, 1989; Laborde, 1982; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1984; Steinberg, Sleeman & Ktorza, 1990; Vergnaud, 1985; Vergnaud, Cortes, & Favre-Artigue, 1987; Wagner, 1981). These difficulties were not surprising for those who tended to think of algebra as requiring the emergence of formal operational thinking. Regardless of one's theoretical stance, the research findings alone seemed to justify delaying algebra instruction until students were about 12 years old and they had already established a solid grounding in arithmetic. To help children overcome the difficulties encountered in this transition from arithmetic to algebra, researchers (e.g., Herscovics & Kieran 1980; Kieran, 1985b) developed teaching approaches to help seventh and eighth grade students use their knowledge of arithmetic to understand algebraic equations.

If adolescents were apparently having so many difficulties with algebra, what could possibly recommend *early* algebra? A quick look at some empirical findings and recommendations can shed light on this matter.

Developmental studies reveal that even seven year-olds understand the basic logic of equations when grounded in reasoning about quantities (Carpenter & Levi, 1998; Schliemann, Brito-Lima, & Santiago, 1992; Schliemann, Carraher, Pendexter & Brizuela, 1998). Additionally, children in elementary school classrooms were found to use algebraic reasoning while they interact with their peers and the teacher to solve relatively complex, open-ended problems (Schifter, 1998). Furthermore, there has been at least one eminently successful attempt to integrate algebra into mathematics teaching and learning beginning from the first grade (Bodanskii, 1991). In Bodanskii's study, fourth graders in the Soviet experimental group used algebra notation to solve verbal problems. These fourth grade students not only performed better than their control peers throughout the school years, but also showed better results in algebra problem-solving when compared to sixth and seventh graders who followed the traditional path of five years of arithmetic instruction followed by algebra instruction from grade six on.

But should we wait until children are 12 years old to focus on the logical relations? One of the main features that characterizes additive structures suggests not. Piaget characterized a structure as a system of relations, stressing that what are important are the relations between elements, and not the elements themselves (Piaget, 1970/1968). Thus, the additive structures correspond to a set of relations between elements, and not only elements, yields, or products. By focusing arithmetic instruction on the latter, we are ignoring one of the inherent characteristics of the number system and of the additive structures domain. Knowledge construction consists "in establishing relations, identifying interactions and constructing interconnections, with which the data provided by experience are organized" (Piaget, in García, 1997, p. 63). Arithmetic instruction typically ignores the relations between elements and their transformations and fails to recognize the basic ways through which we organize experiences and construct knowledge. These arguments further question educational approaches that continue to dichotomize arithmetic and algebra.

In the last fifteen years, some researchers have recommended an algebraic approach during the elementary years of mathematics education. Davis (1985, 1989) argues that preparation for algebra should begin in grade two or three. Vergnaud (1988) suggests that instruction in algebra or pre-algebra should start at the elementary school level so that students will be better equipped to deal later with the epistemological issues involved in the transition from arithmetic to algebra. Kaput (1995) sees the traditional algebra curriculum as alienating students from genuine mathematical experience and as an engine of inequity. He further proposes that the integration of algebraic reasoning across all grades is the key to add coherence, depth, and power to school mathematics, eliminating the late, abrupt, isolated, and superficial high school algebra courses. "An algebrafied K-12 curriculum helps democratize access to powerful ideas." (Kaput, 1995).

The task before educators goes far beyond embracing the general idea of early algebra. For early algebra is not about teaching algebra, as we now teach it, to younger students. It is about new ways of looking at how arithmetic and algebra are interwoven or, as we will attempt to illustrate here, at drawing out the algebraic character of arithmetic. To do this represents a considerable departure from the present state of affairs. How should this be done? What kinds of problems should be used? What sorts of new demands does this place on teachers and students? What sorts of adjustments and new ideas in teaching and learning promise to be fruitful?

Addition and subtraction may provide fruitful contexts for exploring the algebraic nature of arithmetic. This is supported by an impressive series of Soviet studies in mathematics education (e.g., Bodanskii, 1991) in which children discussing, understanding, and dealing with algebraic concepts and relations much earlier than is the norm nowadays. These intriguing developments have helped mathematics educators gradually realize that arithmetic and elementary algebra are far more intertwined than commonly supposed[1].

Here is where the research base for early algebra comes into play. As we noted above, there is already a rich background of research on learning and development that can be drawn upon. One of the underlying themes in research has been that *children's understanding of how quantities combine to make new quantities and break apart into component quantities* may be crucial to much of mathematical learning. We strongly suspect that variable quantities can play an important role in the emergence of

[1] Bodanskii (1991) discusses the tension between arithmetical and algebraic methods for solving verbal problems and concludes, "the algebraic method is the more effective and more 'natural' way of solving problems with the aid of equations in mathematics" (p. 276). We tend to view the issues as demanding a new conceptualization of arithmetic rather than its wholesale displacement by algebra. To be fair, this also seems more in keeping with what Bodanskii and colleagues have tried to achieve.

mathematical variables and that using notation for describing the relations among quantities foreshadows and paves the way for using notation about variables and their interrelations in functional notation. From this viewpoint, addition and subtraction can be approached, from the start as additive functions amenable to description through algebraic notation.

In the present paper we will explore some of the issues young learners face in trying to understand addition and subtraction from an algebraic standpoint. Let us begin to clarify what we mean here by "algebraic standpoint". In many classroom contexts, the expressions, A + B = C and A = C - B, are taken as having different meanings: the first is about adding, the second about subtracting[2]. This focus on arithmetical operations as actions one "does" (that presumably mirror events such as "lost", "won", "used up" in story problems) contrasts sharply with the view that the expressions describe relations among the quantities, A, B, and C. From this latter, algebraic standpoint, the expressions "A + B = C" and "A = C - B" are interchangable; they express the very same relations (albeit in slightly different ways)[3]. But much more is involved than how one parses notation. That said, will be an attempt from here on to defer as much as possible to what children themselves have to say (and write and gesture), for we have found that they often express, in telling ways, the issues they are coming to grips with. Indeed, wherever problems are consistently taken by children in ways differently than they are given by adults, there may be something important for us to learn.

THE RESEARCH SETTING

The present research was undertaken as part of an Early Algebra study of the authors of this paper with a classroom of 18 third grade students at a public elementary school in the Boston area. The school serves a diverse multiethnic and racial community reflected well in the class composition, which included children from South America, Asia, Europe, and North America. We had undertaken the work to understand and document issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting. During the first semester of third grade, when the study here described took place, the teacher had been addressing primarily issues of addition and subtraction. Our activities in the classroom consisted of teaching a two-hour “math class” on a weekly basis, for a total of five weeks. In addition, during the following three weeks we interviewed children about issues related to the content of the lessons. The interviews were treated as an additional source of data as well as an opportunity for the children to develop a greater understanding of problems similar to those presented in class. The topics for the class sessions evolved from a combination of the curriculum content, the teacher's main goals for the semester and the questions the researchers

[2] Indeed, the latter expression may be viewed as altogether incorrect by some children, for whom the “problem” should be on the left side and the “answer” on the “gives” side.

[3] These are not totally exclusive viewpoints. Even an additive comparison problem (Tom has $4.00 more than Maria) is amenable to expression as an implicit or virtual action (e.g., If we were to take away $4.00 from Tom or give $4.00 to Maria, they would have the same amounts).

brought to the table. For example, as the study began, the teacher was reviewing with the children the topics of "more" and "less" and the use of notation for "greater than" and "less than". We were well aware of the relative difficulty of additive comparison problems (Carpenter & Moser, 1982; Vergnaud, 1982) and sensed from the start that additive comparisons would provide a special opportunity for approaching arithmetic from an algebraic standpoint. This approach is similar in several regards to Thompson's (1991) teaching experiments with fifth grade students.

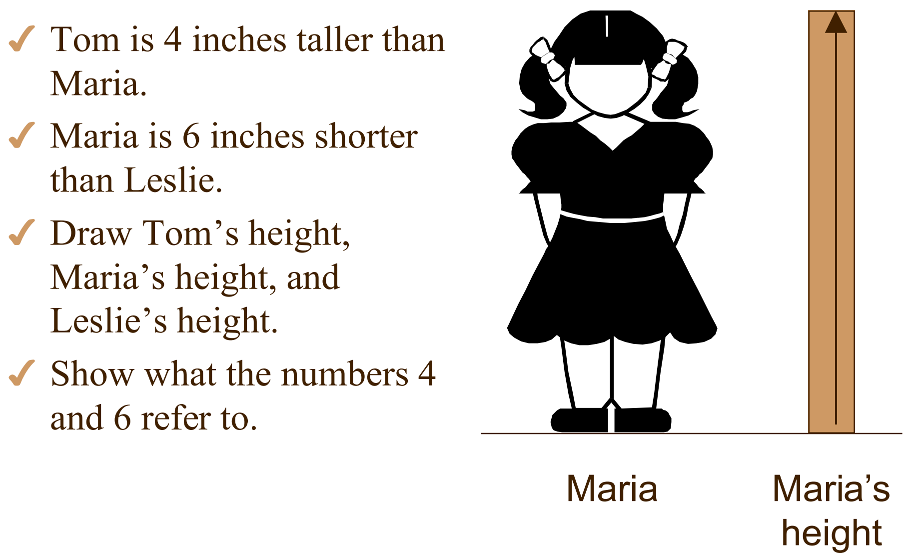
From a research only viewpoint, we would normally wish to follow one or two children in order to trace their thinking over time. But since we were teaching simultaneously as we were doing research, the researchers, who also worked holding cameras, often found themselves divided as helpers in the class activities and as documentarists. For these reasons, it is necessary to tell part of our story with some children, and part with others. We are reasonably confident that the great majority of students were confronting similar issues. Among those issues was that of recognizing, for example, that the problem we focused on entailed differences between heights, which many of the children initially interpreted as heights of individual children in the problem.

THE PROBLEM

Here is the problem we devised for the fourth week's class. We had already spent three weeks working with vector-like representations for quantities similar to the directed line segment next to Maria. In the first two weeks they showed us that they spontaneously represented amounts through iconic representations (E.g., candies drawn in paper wrappers, fish as icons of fish, with eyes, fins, scales, etc.), and seemed to prefer them to our suggestions to use vectors. Gradually, however, they began to use "arrows" to represent both discrete and continuous quantities and would with a certain delight shift discussion from one context to the other (e.g., fish dying in a fishbowl, to spending money or consuming cookies) for the same configuration of arrows. This, incidentally, attests somewhat to the usefulness of generic referents for quantities over iconic representations, which are not easily interchangeable.

At the beginning of the fourth class, the following problem was shown to the children via a projector onto a large screen. Some of the students read the sentences aloud as the first author read it to them.

Figure 1: A problem of additive differences



QUANTITIES AND THEIR DIFFERENCES

The children clearly understood that the problem referred to the heights of three children, and that there was some variation in these heights. However, it was apparent that many of the students were thinking about the numbers in ways quite differently from us. Claudia decided that Maria was taller than Tom and then changed her mind. She and her classmates were treating the numbers as total heights. ("Tom is four inches tall…"). Nathia decided that Maria must be six inches tall and Leslie was 12 inches tall. Michael volunteered that Tom was 10 inches tall (see Transcript 1 and Clip 1 below), apparently having added the two numbers given in the problem. In order to make sure the children were clear about the magnitude of the measures, we pulled out a measuring tape and showed them how wide an interval 4 inches would span. Most children immediately tried to separate their two palms by that amount, as if to calibrate the amount in a personally meaningful way. However, although children recognized that 10 inches was inordinately short for a child, they continued to view the measures in the problem as referring to absolute heights as shown in the discussion with Michael (Transcript 1).

Transcript 1: Michael affirms his initial reading of the problem. Many other students appear to agree with him.

10 INCHES TALL

David: … [Tom] must be about this big right?

Michael:	[Yeah..] no, Tom, Tom is…: Ten inch…
David:	… must be a puppet. Then why did you, why did you blurt out, why did you say very quickly that Tom was ten inches? I have a feeling I know why you said it.
Michael:	Maria… She is six inches…
David:	It says Maria is six inches tall?
Michael:	But I don't mean she's this small.
David:	No? Well, what does it say there, Michael? Read it. Read the second sentence for us.
Michael:	Maria is six inches shorter than Leslie.
David:	Than Leslie, right. So, is anybody six inches tall in this problem?
Michael:	Yes.
David:	Who?
Michael:	Maria. (others agree)
David:	Maria? She's six inches tall? Is that what the problem says?
Michael:	Yeah…

Clip 1: Michael trying to show a height of 6 inches (what he takes as Maria's height).

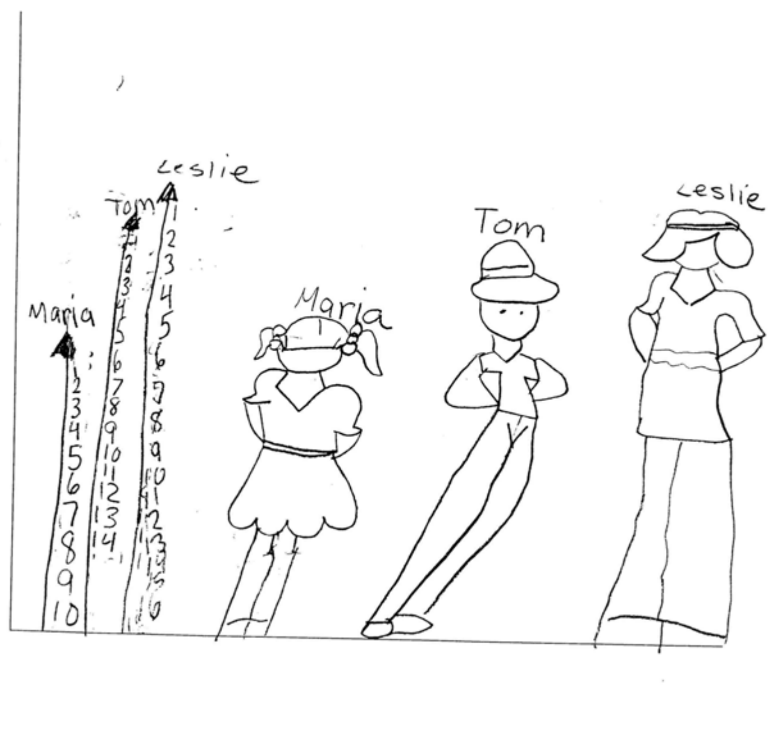
We then decided to enact the problem in front of the class with three volunteers. Jessica was chosen to represent Tom. With some discussion, the children agreed that the stand-in for Maria (Jennifer) should be shorter than Jessica (Tom). Finally, Reynold was chosen to represent Leslie; he was noticeably taller than the other two. In the course of this acting out, the three actors found it helpful to use their hands as we reread the problem and verified whether we had the information straight. We then asked the students to show the differences in height—e.g. "how much taller is Tom than Maria." Faced with this question, the children sometimes pointed to one child, sometimes to the other. At our insistence to show "how much more," they preferred to show one height and then the other by placing a hand on each respective head, either in sequence or simultaneously. In retrospect, it appears that the children were not inclined to view the relative differences in heights as entities that occupied a region in space. It is as if "more" was not a bona fide amount, but rather a state of affairs that required reference to two legitimate amounts, namely the two children referred to in the comparison. However, Jennifer clearly identified an interval above her head to represent how much taller "Tom" was than her.

In this acting out, the members of the class became convinced that they now had the order of the heights straight. And indeed they did. Leslie was the tallest; Maria was the shortest, Tom was in between. When asked the trick question of whether Tom was "taller or shorter?" some students voiced support of one position, others supported the other. Then they quickly realized that Tom was taller (than Maria) and shorter (than Leslie) at the same time.

DRAWING THE CHILDREN'S HEIGHTS

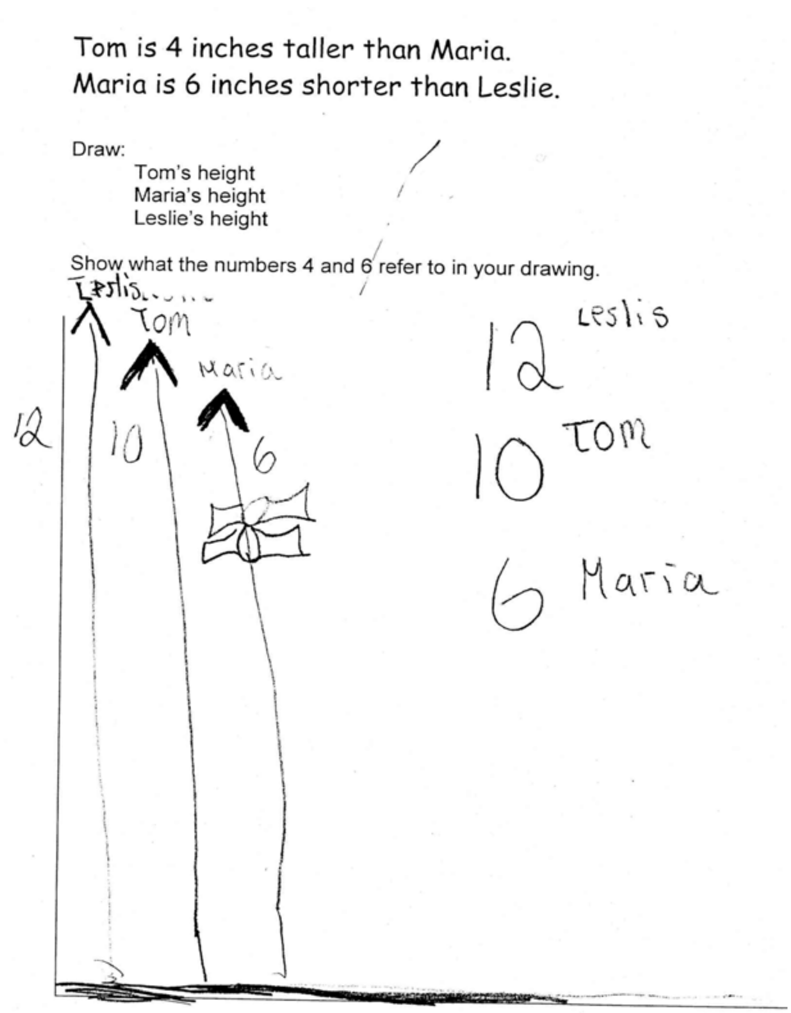
Now that the children were fairly comfortable with the premises of the problem, we asked them to draw the heights of the three protagonists on a piece of paper using "arrows." As mentioned before, they had already practiced using vector-like symbols ("arrows") in earlier sessions. We were curious about how they would include the values, 4 inches and 6 inches, in their drawings. Since we had collectively settled the issues of the relative heights, not surprisingly 15 of the 16 children in the class that day who made drawings got the orders of the heights correct. Even though they openly compared and discussed their drawings, there was a fair degree of variability in their approaches. In a continuum from iconic to use of arrows in a comparative fashion, two children produced iconic drawings of Maria, Tom, and Leslie (with hands, feet hair, etc.) and 4 more, like Samantha, drew both arrow diagrams and iconic pictures (See Figure 2).

Figure 2: Samantha's interpretation of the student's heights. The instructions, the same as in Fig. 3, are not shown.



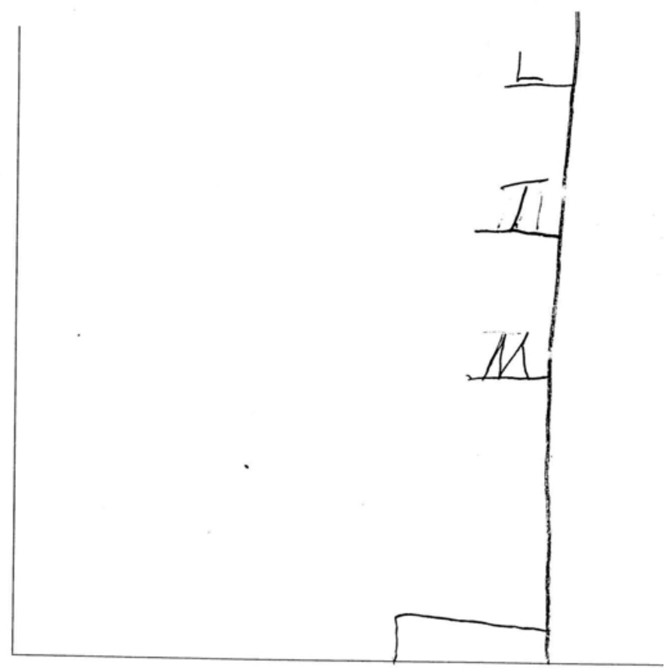
Two other children drew a person-arrow hybrid, that is, vertical lines with a head on top; 8 children simply drew arrows to represent the children or their heights as in Melissa's case (Figure 3).

Figure 3: Melissa's rendering of the student's heights.



Yasmeen drew one vertical line with the letters T, L, and M labeling distinct heights along the line (see Figure 4). Incidentally, this child hardly ever participates in classroom discussion and tends to work in isolation, not always following the prescribed activities.

Figure 4: Yasmeen's rendering of the student's heights.



Twelve of the 16 pupils used numbers in their drawings. Although only 2 numbers were given in the problem, every student who used numbers provided three, one each for Tom, Maria, and Leslie. (Actually, in two cases children drew a series of numbers next to the children's height-arrows as shown in Samantha's drawing (Figure 2). In 7 of the 12 cases where numbers were used, children provided answers that were consistent with the information given in the problem. Interestingly, however, none of our students spontaneously used the values 4 inches and 6 inches explicitly in their drawings. It is not that they did not make use of this information; as we said, 7 children gave total heights that embodied these differences. But they did not label any parts of heights or intervals between heights as corresponding to these values.

It is also interesting to note that the greatest height given by a student was 16 inches. Since in the course of our earlier discussion we had already noted that the children in the class were in the neighborhood of 48 inches tall, I asked them to explain why the numbers were all so small. Melissa wittingly volunteered that Tom, Maria, and Leslie were probably 3 dolls!

Kevin's dialogue with Barbara shows how he worked out the differences in height between Leslie and Tom and, when prompted, how he attempted to hypothesize a height for Maria.

Transcript 2: Kevin deduces the difference between Leslie and Tom's heights. He has speculated about Maria's height.

RELATIVE HEIGHTS

Bárbara:	So show us, …
Kevin:	So she's (Leslie) taller than Tom by two inches. Mmhumm… cos Tom's taller than Maria by four inches… Mmhumm… and Maria is taller than, and Leslie is taller than Maria by six inches so that means that if he's taller than her by four inches and she, then Leslie has to be taller than Tom by two inches.
Bárbara:	Mm… and to figure that out do you need to know exactly how high each one is?
Kevin:	Yes
Bárbara:	You do? How did you do it if you don't know how high they are?
Kevin:	I do... cuz Maria's... Maria's…
Bárbara:	Tell me exactly how high they are.
Kevin:	Maria's... four feet six inches… and uh…
Bárbara:	Where did you get that from?
Kevin:	I don't know it just looks like that.
Bárbara:	I think, I think you're just inventing that.
Kevin:	I am.
Bárbara:	You are.

Clip 2: Kevin explaining his logic.

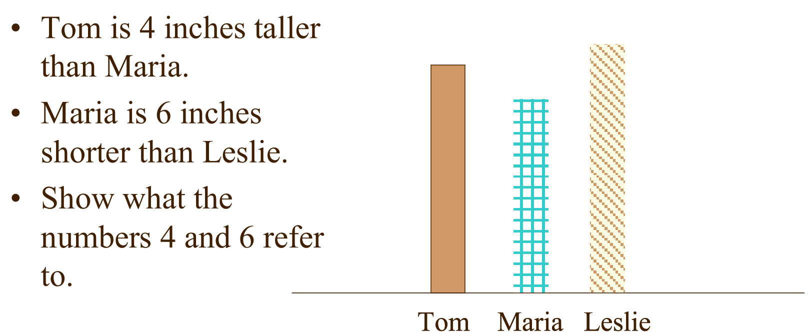
INFERRING AND VISUALLY INDICATING THE DIFFERENCES

After the students had prepared their own drawings, we showed the diagram in Figure 5 and asked them to explain their answers. Typically, a student would point to the respective rectangle and read off the height. Children seemed to treat their answers as “the correct” answers rather than one correct set among many possibilities. When asked how they knew Leslie was 12 inches tall, they would justify their answers by referring to the relative information (“because Maria is 6 inches shorter”). Apparently, none of the children realized that neither child’s height was determined by the information given in the problem.

It was revealing how children attempted to relate the numbers 4 and 6 to the diagram, when prompted. We expected that since they had often used the numbers correctly in their drawings, they would easily isolate the parts of heights by which one exceeded the other or fell short of the other. However, this did not occur. As they had done before when comparing actual children in the room, they tended to point to the highest part of one or both “children”, i.e., the rectangles in the diagram. When we continued asking them to show us something that corresponded to 4 inches, they changed from marking off a vertical line within a rectangle to sweeping out a region.

Some children would highlight the region between two rectangles. Others would refer to a vertical region. (A more careful analysis of the videotape will be provided in the final version of this paper).

Figure 5: Diagram for a short whole class discussion.



With encouragement, several students came to the realization that we could label vertical regions or parts of rectangles as corresponding to 4 and 6 inches. This discovery seemed to be crucial in the children's coming to understand differences as quantities that can be visually expressed in a consistent manner. Here is how Sarah shows 6 as the difference between two heights:

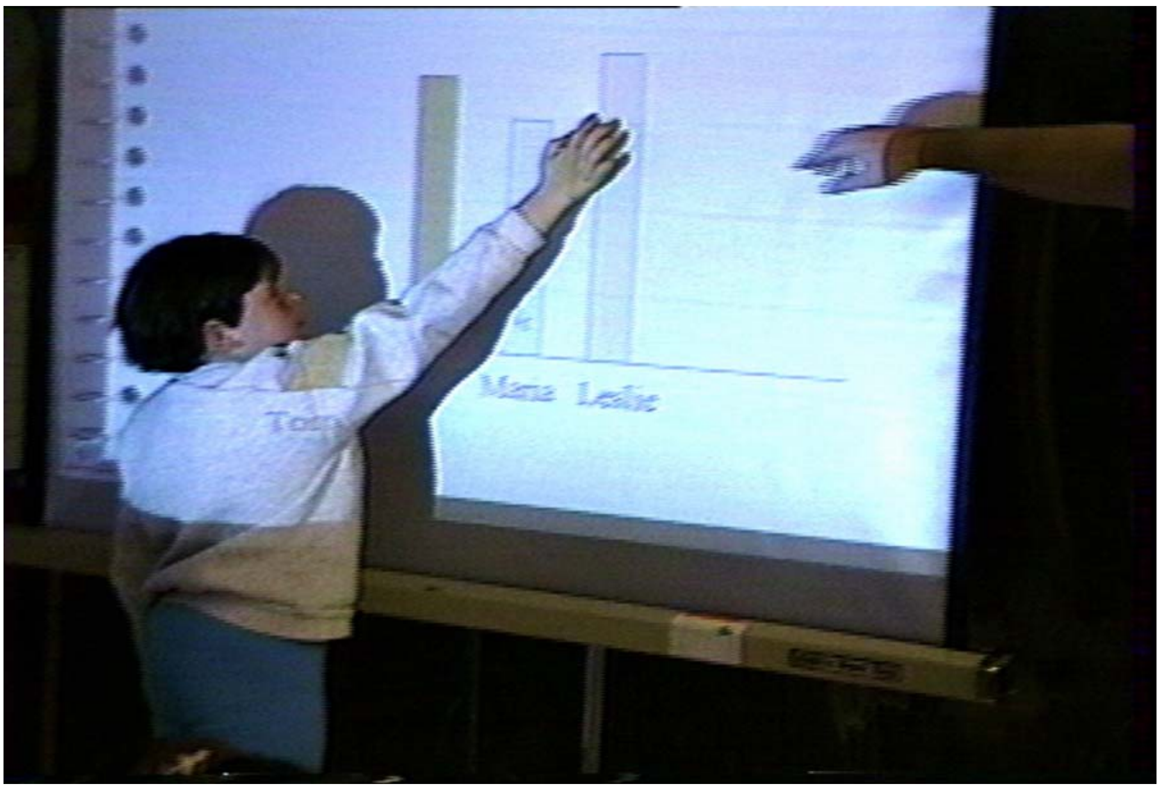
Transcript 3: Sarah points to an additive difference..

WHERE IS THE 6?

David:	She's (Maria) six inches shorter... Where does that six go? If you had to point to something up here (indicating projection of Figure 5 on the screen), Sarah, could you show me what six refers to? …Go ahead. (To the other students) Take a look at what Sarah's going to do now.
Sarah:	(Pointing to the difference between Maria and Leslie) This space right here
David:	See this space right here. That's kind of the difference…

Actually, Sarah's gesture has three parts. First she sweeps her hand from the top of Maria's height (line segment) horizontally rightward to the corresponding height against Leslie's line segment. Then she sweeps her hand upwards indicating the region by which Leslie's height surpasses that of Maria. Finally, she sweeps out a diagonal from the top of Leslie's height to the top of Maria's height. The three movements comprise a triangle. In accepting her answer, David highlights the second of the two movements. Incidentally, the same question sometimes evinced slightly different responses from other students. Erica, for example, restricts herself to the first of Sarah's three movements to indicate where the "6" is. Other students make only the diagonal movement. It is not altogether clear what these individual variations mean but together they suggest that some students feel that it is necessary to include both elements (each child's height) in expressing the difference.

Clip 3: Sarah indicates the region of Leslie's height that corresponds to the amount by which she surpasses Maria (6 inches).



We then handed each student Table 1, shown below, and asked them to figure out the heights of Maria and Leslie in “Story 1” namely, a case in which Tom was allegedly 34 inches tall. (Actually, the value T-4 was not explicitly written into the Table but we helped several of the students with this part so that they had at least two examples to work with; their job was then to determine the notation for Leslie in Story 4.) The table itself was possibly a new form of representing mathematical information for the children and so it is not surprising to us that they would need a little guidance in understanding what sort of information the table presented and what sort of information needed to be filled in. Within a few minutes they were relating the information in Story 1 to the information given before and most students were able to determine heights consistent with the relative information.

One might wonder why we did not provide the table at the beginning of the class so that the children could work with specific values. In our earlier work, however, we had noticed that the students were often inclined to perform hasty numerical computations without reflecting upon the relations among the quantities at hand. By initially treating the heights as unknowns, we had been able to contain their urge (somewhat) to compute before thinking about the relations among the quantities. Now that we had achieved our basic purposes, we turned to the instantiated values as a means of varying the quantities while maintaining certain properties (the differences in heights) invariant. We named the table “Table of Possible Values” to emphasize the fact that we were dealing with hypothetical values. This, we hoped, would reduce their puzzlement, were they to have tried to reason out story 2 having previously taken the values of story 1 as "the true values".

From a certain point of view, the stories contradict each other. For example, if story 1 has correct information, then the information in story 2 cannot simultaneously be correct, and so on. Getting this straight is part of the challenge of thinking hypothetically and thinking about the quantities as having variable quantities.

Figure 6: Four "stories" about the children's heights.

What if…			
Story 1			

Most children had time to complete only the first two stories. But that was enough for them to realize that the initial problem information could involve more than one set of possibilities, something that had not been clear when they were trying to make their drawings or enactments of the heights. When we later discussed the answers in the table in front of the whole class, we tried to see whether they appreciated this by asking, “Which story is true?” By then all seemed to be comfortable with the idea that none was necessarily true. Melissa drew attention to the hypothetical nature of the task by pointing out that the table “says what if….”

Kevin worked through each of the examples with no difficulty. The value, T-4, was worked out with some scaffolding from one of the researchers. The result, T+2, was his own production. His explanation of "T+2" follows in transcript and clip 4, below.

Figure 7: Kevin's working out of various scenarios (stories), each consistent with the original information.

Tom is 4 inches taller than Maria.
Maria is 6 inches shorter than Leslie.

Fill out the table for story 1.
Imagine Tom is 34 inches tall. How tall will Maria and Leslie be?

What if...	Tom	Maria	Leslie
Story 1	34 in.	30 in.	36 in.
Story 2	37 in.	33 in.	39 in.
Story 3	41 in.	37 in.	43 in.
Story 4	35 in.	31 in.	37 in.
Story 5	T	T-4	t+2

Then work out the answers for stories 2, 3, and 4.

Transcript 4: An explanation of "T+2"

TALL + 2 INCHES

Kevin: Oh, OK, Tall.
Tom's taller than Maria by 4 inches,
and Tall plus two equals…
Leslie's taller than Tom by two inches (writing T+2 for Leslie).

Clip 4

In a private discussion, Nathia and Jennifer were also able to conclude that Leslie's height must be T+2. When they attempted to explain their reasoning to the rest of the class they suggested that the letter T under Tom could stand for "**t**all" or "**t**en (inches)". We mentioned that we were actually using it to stand for "whatever Tom's height was." Given the fact that the class was ending it would have been pointless to require that all students understood how letters represented variable quantities in Story 4. We were satisfied for the moment that we had been able to initiate discussion of what would be a gradual familiarization with variables (here in the sense of multiple possible values) and how they are expressed.

ALGEBRAIC NOTATION BECOMES LESS MYSTERIOUS

Approximately one month later we interviewed each of the children regarding relative differences problems. By and large, we were surprised by how they had become familiar and confident with such problems. The following discussion with Jennifer and Melissa followed an example quite similar to the one discussed in class (about relative heights). Melissa's characterization of x as standing for a "surprise number" suggests that they are thinking about letters as standing for a single, determined value. This is consistent with Jennifer's point that "everyone in the world has a height". However, at the end of the session they seemed to accept the idea that x could stand for each and every value (of money) that we had worked with over the session. Could it be that thinking in terms of unknowns is a natural stepping stone to thinking in terms of variables (as we suspect) or are these fundamentally different ways of thinking?

Transcript 5: Melissa and Jennifer discussing the meaning of x (one month after the class referred to).

SURPRISE NUMBERS

David:	...x's there? What do you, what do you think that was about? I mean, why were we saying, why were we using x's?
Jennifer:	You might think like the x is Alan's height and it could be any height.
Melissa:	Uh huh.
David:	So it could be any height?
Melissa:	Yeah.
David:	And... and what about Martha's height?
Jennifer:	It has to be 3 more than Alan's.
David:	OK. So like if x is 40...
Melissa:	Yeah.
David:	What would Martha's height be?
Both:	43.
David:	OK. If x was a hundred...
Jennifer:	It would be a hundred and three.
David:	OK, now, if I *didn't* know Alan's height, and I just had to say, "Well, I don't know it so I'll just call it 'x'…"
Melissa:	You could guess it.
Jennifer:	You could say like, well it would just tell you to say any number.
David:	Why don't I use an x and say whatever it is I'll just call it x? (umm) Do you like that idea, or does that feel strange?
Jennifer:	(Melissa seems to accept it but Jennifer does not) No, that’s strange.
David:	It feels strange?
Jennifer:	Cos it has to, it has to have a number. Cos... Everybody in the world has a height.
David:	Oh... OK. Well, what if I tell you, I'll do it a little differently. I have a little bit of money in my pocket, OK, do you have any coins, like a nickel or something like that?

Jennifer:	All mine's in the bag...
David:	OK, I'll tell you what: I'll take out, I'll take out a nickel here, ok. And I'll give it to you for now. I've got some money in here (in a wallet) can we call that <u>x</u>? (hmm) Because, whatever it is, it's (inaudible)...
Jennifer:	You can't call it <u>x</u> because it has... if it has some money in there, you can't just call it <u>x</u> because you have to count how many money is in there.
David:	But what if you don't know?
Jennifer:	You open it up.
David:	Because if you tell me its like 50 cents… yeah, but we haven't opened it yet. We will later.
Melissa:	So its like a surprise.
David:	Yeah. It's like a surprise. So let's call the <u>x</u> in this case the amount of money that I have in here, OK?
Melissa:	So its like an <u>x</u>, and then…
David:	Now how much money do I…
Jennifer:	<u>X</u> is probably like a nickel.
David:	How much do I have here?
Both:	<u>x</u> 5.
Jennifer:	5 cents.
David:	<u>x</u> 5 or...?
Melissa:	<u>x</u> five...
David:	<u>x</u> five.
Jennifer:	<u>x</u> **plus** 5
David:	She's [Jennifer's] saying it a little differently. If I put this back in here, now I have <u>x…</u>?
Melissa:	<u>x</u>.
David:	No, I had <u>x</u> before. And now I'm putting it in here and I've got...
Both:	<u>x</u> **plus** 5.
David:	<u>x</u> plus 5, sure. And that…
Jennifer:	The amount of money in there is… *any* money in there. And after..., if you like add five, if it was like... imagine if it was 50 cents, add five more and it would be 55 cents.

Clip 5: Melissa and Jennifer discussing the meaning of x.

LINKS TO EARLY ALGEBRA

We took as our goal to illustrate how problems normally viewed as sitting squarely in the domain of arithmetic can take on an algebraic character from the very start. This argument has been made, in slightly different terms, by other members of the present symposium. We find ourselves in agreement with Kaput and Blanton's (1999) appeal for an algebrafying of the arithmetic curriculum. The move for Early Algebra would seem to represent a new way of looking at how arithmetic is taught and learned. As such, it is less about displacing curricula by new topics than in looking at time-honored topics in a new light and with a new set of attitudes. We expect that this will require a lot of investment in teacher development and, as we tried to suggest in the present paper, an important role for classroom research that has both an applied and basic character, where researchers and practitioners work closely together. This latter idea has been echoed by all the contributors to this symposium at one time or another as well as our colleagues in mathematics education.

The CGI (Carpenter and Levi, 1991) work presented here highlights how much can be achieved by focusing on written notation itself and the playful ways in which children delight in finding written expressions that capture generalizations they have noticed about numbers. For those of us who have often looked upon symbol manipulation as inherently meaningless activity, it leaves a lot to ponder about. Written symbols are at times meaningless, but at other times, and apparently from a very early age, they may be treated as conceptual objects, the meaning of which derives in part from children's knowledge of number and in part from their sense of syntactical and logical coherence. Any attempt to evoke the algebraic character of arithmetic will have to consider the fundamental role of natural and written language, both of which are symbol systems.

As we prepared this report we came across a notable, earlier work by the discussant (Thompson, 1991) that shares several of the premises of our own work and reveals some similar findings. In short, Thompson found that 5^{th} grade students were still struggling with some types of additive differences. The problems he used, with the exception of some given on day 1, were admittedly more complex than those used in our study; many entailed second-order additive differences (differences between differences). Nonetheless, the mistaking of, for example, a second-order difference (how much more brother A towers over his sister than brother B towers over his sister) with a first order difference (how much taller brother A is than sister A) echos the kinds of confusions our students exhibited when first presented with word problems.

We suspect that such difficulties are only partly about "reading carefully". As we noted in Clip 1, students continued to insist that Maria was 6 inches tall even though they very carefully read that "Maria is 6 inches shorter than Leslie". It seems that parsing the writing notation is part of a larger story. This larger story seems to involve students' conceptualizations about how quantities can be formed from and broken apart from other quantities. This sort of analysis suggests that we we also take into account non-linguistic, spatial representations as part of the symbolic repertoire that comes into play in the emergence of early algebra. This is where other representations of a particularly visual character, tables, vector diagrams, and free-form drawings take on importance. If algebraic knowledge ultimately rests upon intuitions and knowledge about physical quantities--if for example, variable quantities are a gateway to algebraic variables--then we will need studies documenting the issues that children are dealing with. There is already a good deal of evidence that a mismatch between operations on quantities and written notation lies at the heart of many of students' difficulties with multiplicative structures and rational number concepts (e.g., Carraher, 1996). Our work with third graders suggests that there are similar issues to be worked out regarding the mapping between written notation, including notation for equations and for numbers themselves, and their sense of what is implicit regarding the relations among quantities.

The key to viewing arithmetic through the lens of algebra consists in recognizing the significance of physical quantities (and actions and events with quantities) in children's thinking as forerunners for the concepts of mathematical variable and functions.

Children commonly work with unknown quantities in arithmetic but largely as the desired end point of a sequence of computations. Traditional wisdom holds that a problem should *contain* the operations, which, if performed, will lead to an answer for the desired unknown value. However, this only works for the most contrived and stereotyped of situations (which typical grade school textbooks abound in). A simple statement such as “Tom is 4 inches taller than Maria” suggests to many children the operation of adding 4, yet Maria’s height ought to be *less* than Tom’s height. The additional fact that Tom’s height is not given can perplex students expecting to encounter information in the problem given in the order that it need take in the computational solution. A problem such as this requires far more than simply inverting operations, e.g., using subtraction rather than addition. It requires taking a strikingly different view of mathematical expressions. To the child heavily schooled in arithmetic as computation, a plus sign evokes the physical action of adding, giving, joining; a minus sign evokes the idea of losing, spending, diminishing (Carpenter & Moser, 1982). There is nothing inherently wrong with this belief, and surely children are going to begin from such beliefs, but they will not get very far in the vast sea of problems for which no such physical actions can be easily identified (see, for example, Meira, 199x). The simple idea of a comparison ("Sally is 5 inches taller than Jamie") provides a case in point. As we noted, young children can, with encouragement, use continuous visual diagrams to represent the relative heights of children. But a problem of this sort requires isolating a part or interval of these drawn quantities as representing the additive differences between two children's heights. This requires treating the height difference as distinct quantity in its own right. The job of identifying this new conceptual entity can be approached through attempting to draw it. But children must also be able to talk about such relational entities. As the brilliant work by Bodanskii (1991) and others has shown, this is one place where algebraic notation can play an important role. If x is the difference between quantities A and B, and B is greater than A, then it is true that $A = B-x$. It is also true that $A + \mathbf{x} = B$. And x is equal to B-A. From a traditional arithmetical standpoint, these are three different statements telling us to "do" different computations. From an algebraic viewpoint, they are merely different ways of describing the same relations.

Algebra knowledge is not always grounded in thinking about quantities. As Kaput (1995) has noted, algebra is not always about generalizing and formalizing patterns and constraints; there is a very legitimate sense in which algebra can be viewed as the syntactically-guided manipulation of formalisms. And as Resnick (1985) noted, there is a point where one can forget about the situations that gave rise to the algebra and extend one's knowledge within the rules of the algebraic symbolic system without having to return to the situations that originally gave meaning to the expressions.

There is good reason to honor the goal of having students treat algebra as a self-contained system for manipulating symbols and deriving implications from axioms, givens and rules of inference. But this is certainly not an ideal point of departure. For most students, what they know about physical quantities and their interrelations will provide a rich starting point for algebraic understanding. We can help by seeing bringing out the algebraic character of arithmetic and helping familiarizing students with some of the representational tools of algebra well—notation, tables, and diagrams—well before algebra enters the curriculum as a formal topic of study.

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