Bringing Out the Algebraic Character of Arithmetic: Instantiating Variables in Addition and Subtraction[1]

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We report findings from a one-year teaching experiment designed to document and help nurture the early algebraic development of third grade students. We focus on an arithmetic problem that fixes some measures but allows more than one solution set. We highlight how children dealt with the fact that the quantitative relations referred to particular measures on one hand (and in that sense were arithmetical), and were meant to express general properties not bound to particular values, on the other (and in this sense were algebraic). We look at the role of instantiated variables in this tension and transition between the particular and the general.

After carefully documenting the difficulties of algebra students (Booth, 1984; Da Rocha Falcão, 1992; Filloy & Rojano, 1989; Kieran, 1985, 1989; Laborde, 1982; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1984; Steinberg, Sleeman & Ktorza, 1990; Vergnaud, Cortes, & Favre-Artigue, 1987), the field of mathematics education has gradually come to embrace the idea that algebra need not be postponed until adolescence (Bodanskii, 1991; Davis, 1985, 1989; Kaput, 1995; Vergnaud, 1988). Researchers have increasingly come to recognize that young children can understand mathematical concepts assumed to be fundamental to learning algebra (Brito Lima & da Rocha Falcão, 1997; Carraher, Schliemann, & Brizuela, 1999; Schifter, 1998; Schliemann, Carraher, Pendexter, & Brizuela, 1998). Many researchers and educators now believe that elementary algebraic ideas and notation may play an important role in students' understanding of early mathematics. To support this change in thinking and practice the field needs research on how young learners reason about problems of an algebraic or generalized nature.

The present research was undertaken as part of an early algebra study of the authors of this paper with a classroom of 18 third grade students at a public elementary school in the Boston area during a one-year teaching experiment. The school serves a diverse multiethnic and racial community reflected well in the class composition, which included children from South America, Asia, Europe, and North America. We had undertaken the work to understand and document issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting. Our activities in the classroom consisted of teaching a

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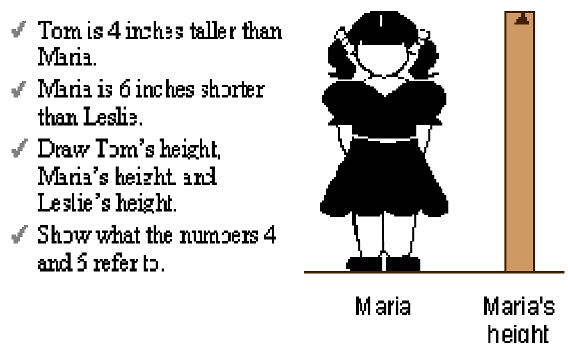
two-hour "math class" on a bi-weekly basis. The topics for the class sessions evolved from a combination of the curriculum content, the teacher's main goals for each semester, and the questions we brought to the table. We were well aware of the relative difficulty of additive comparison problems (Carpenter & Moser, 1983; Vergnaud, 1982) and seized upon additive comparisons as a special opportunity for approaching arithmetic from an algebraic standpoint.

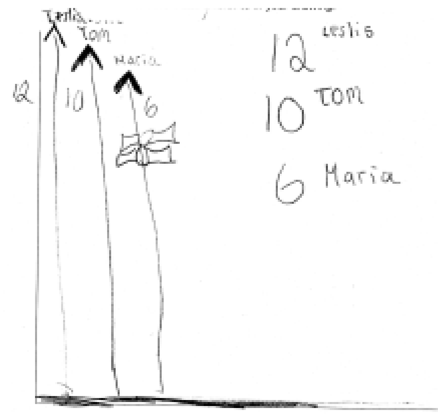
At the beginning of the fourth class, the problem below was shown to the children via projection onto a large screen. The children clearly understood that the problem entailed the heights of three children. However, many of the students were apparently treating the numbers as total heights. ("Tom is four inches tall…").



One child decided that Maria must be six inches tall and Leslie was 12 inches tall. Another volunteered that Tom was 10 inches tall, after adding 6 plus four, the two numbers given in the problem–although he was puzzled by how a child could be so short. The problem gave information about differences between heights; the children treated the difference information as actual heights. (This shift in level of discourse calls to mind what Thompson [1991] noted among fifth grade students, who mistakenly treated second-order differences as first-order differences.) As an attempt to get the children thinking about differences between heights, we decided to enact the problem in front of the class with three volunteers. In the course of this acting out, the three actors (students representing the people in the problem) helped the class gradually recognize the correct order of the children in the problem.

When the pupils had established a consensus about the relative order of heights of the three protagonists, we asked them to provide us with notations showing the information from the problem, and to try to indicate where the numbers 4 and 6 should "go" in their diagrams.

When they were asked to represent the children in a drawing, 12 of the students assigned a value, in inches, to each member of the story. The diagram below typifies this. Note that 6 is taken to be Maria's height; but there is also a difference of 6 inches between Maria's and Leslie's heights. So this solution treats the numbers both as referring to actual heights as well as respecting the relative values.



Although only two numbers were given in the problem, every student who used numbers provided three, one each for Tom, Maria, and Leslie. (Actually, in two cases, children drew a series of numbers next to the children's height-arrows.) In 7 of the 12 cases where numbers were used, children provided heights for Leslie, Tom, and Maria that were consistent with the numbers given in the problem. But they did not label any parts of heights or intervals between heights as corresponding to these values. Even as they eventually came to realize that the measures, 4 in. and 6 in. referred to something other than total heights, many remained hesitant about what region of their diagram corresponded to these values[2].

In the following classroom example Kevin explains how he determined the heights.

> Kevin: So she's (Leslie) taller than Tom by two inches. Mmhumm… cos Tom's taller than Maria by four inches… Mmhumm… and Maria is taller than, and Leslie is taller than Maria by six inches so that means that if he's taller than her by four inches and she, then Leslie has to be taller than Tom by two inches.
>
> Bárbara: Mm… and to figure that out do you need to know exactly how high each one is?
>
> Kevin: Yes
>
> Bárbara: You do? How did you do it if you don't know how high they are?
>
> Kevin: I do... cuz Maria's... Maria's…
>
> Bárbara: Tell me exactly how high they are.
>
> Kevin: Maria's... four feet six inches… and uh…
>
> Bárbara: Where did you get that from?
>
> Kevin: I don't know it just looks like that.
>
> Bárbara: I think, I think you're just inventing that.

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[2] Adults understand that 4 inches corresponds to that part of Tom's height that surpasses Maria's height (ie. lies above her head). Even when the children understood that 4 inches was "how much bigger" Tom was than Maria, they tended to indicate where this value lay (4 in.) by pointing to the top of each head. They did likewise when showing the "difference" between their own classmates' heights. For many, a difference could not be shown as an interval along a spatial dimension; one had to re-enact the action of comparing!

Kevin: I am.

Kevin instantiated Maria's height to 4 ft. 6 in. (was this another way to get the numbers 4 and 6 into the problem?) and then generated values for the other children consistent with the information about the relations among the heights.

Once the children distinguished between heights and their relative differences, we asked them to consider several scenarios or "stories". In "story 1" Tom's height was supposedly 34 inches; in story 2 his height was instead 37 inches; Maria and Leslie were each assigned heights of 37 inches in stories 3 and 4, respectively. In the final story Tom's height was listed simply as "T", which *we* took to refer to Tom's height *whatever* it might be. As far as we can tell, this was the first occasion in which the children were shown letters standing for unknown measures. The students quickly caught on to the idea that the first four stories were mutually exclusive: i.e., at most, one of them could be true. Within a few minutes they were able to determine heights consistent with the relative information provided at the beginning of the class. Our attention then turned to story 5, which, in *our* eyes, was a generalized description of each of the previous stories.

Kevin filled in the table (see below), concluding that in each story Maria had 4 inches less than Tom. In the final case, where Tom's height was given as T, after some prodding from the interviewer, he noted that in each story Maria was 4 in. shorter than Tom, which he expressed as "T-4". Kevin then concluded that Leslie' s height could be described by "T+2". His explanation shows him trying to find words for this new notation (note, for instance, the interpretation for T as "tall").

Kevin: Oh, OK, Tall. Tom's taller than Maria by 4 inches, and tall plus two equals…

Leslie's taller than Tom by two inches (writing T+2 for Leslie).

Two other children correctly solved the general case but, when asked what T stood for, gave "tall" and "ten" as possible interpretations. We tried to explain that T could stand for "whatever Tom's height might be". This did not settle the issue, but over the next several classes we found students increasingly using the words "whatever" and "any" to explain the meaning of letters representing measures.

Tom is 4 inches taller than Maria.
Maria is 6 inches shorter than Leslie.

Fill out the table for story 1.
Imagine Tom is 34 inches tall. How tall will Maria and Leslie be?

What if...	Tom	Maria	Leslie
Story 1	34 in.	30 in.	36 in.
Story 2	37 in.	33 in.	39 in.
Story 3	41 in.	37 in.	43 in.
Story 4	35 in.	31 in.	37 in.
Story 5	T	T-4	t+2

Then work out the answers for stories 2, 3, and 4.

In the sixth and seventh weeks we interviewed children about issues related to the content of the different lessons. The interviews served as an additional source of data as well as an opportunity for the children to develop a greater understanding of problems similar to those presented in class. By then the students had become noticeably familiar with additive differences. Their understanding of unknowns and variables was still under development, as the following interview excerpt reveals. We enter the interview precisely when two pupils, Jennifer and Melissa, are explaining what "x+3" means. The context was a height problem in which Martha was said to be three inches taller than Alan.

> David (interviewer): Why were we using *x'*s?
>
> Jennifer: You might think like the *x* is Alan's height and it could be *any* height.

This would seem to suggest that Jennifer has settled the basic idea of what a variable is. But shortly thereafter it becomes clear that there are still matters to consider.

> David: OK, now, if I *didn't* know Alan's height, and I just had to say, "Well, I don't know it so I'll just call it '*x*'…."
>
> Melissa: You could guess it.
>
> Jennifer: You could say like, well it would just tell you to say *any* number.
>
> David: Why don't I use an *x* and say whatever it is I'll just call it *x*? (umm)
>
> Both: (puzzlement)
>
> David: Do you like that idea, or does that feel strange?

Jennifer: It feels strange.

Melissa: No, I pretty much…[unlike her classmate, she is comfortable with using x in this manner.]

David (addressing Jennifer): It feels strange?

Jennifer: Yes, 'cause it has to, it has to have a number. 'Cause.… **Everybody in the world has a height**.

Jennifer seems to think that it would be inappropriate to use the letter *x*, representing *any* height, to describe Alan, since Alan cannot have *any* height; he must have a particular height. So the interviewer turns the discussion to another context to see if Jennifer's reaction extends beyond the particular case at hand.

David: Oh.… OK. Well,…I'll do it a little differently. I have a little bit of money in my pocket, OK–(to Jennifer) do you have any coins, like a nickel or something like that?

Jennifer: All mine's in the bag [in the classroom]...

David: OK, I'll tell you what: I'll take out, I'll take out a nickel here, OK. And I'll give that to you for now. I've got some money in here [in a wallet] can we call that x? (hmm) Because, *whatever it is*, it's that, it's the amount of money that I have.

Jennifer: **You can't call it x because it has... if it has some money in there, you can't just call it *x* because you have to count how many money [is] in there**.

David: But what if you don't know?

Jennifer: You open it and count it.

Jennifer insists that it would be improper to refer to the money in the wallet as *x*, since it holds a particular amount of money. (Melissa, who likens the value of x to "a surprise", experiences little, if any, conflict; *x* is simply a particular value that one does not know.) In a sense, Jennifer finds the example inconsistent with the idea of *x* as a variable, that is, something capable of taking on a range of values. And she has a point: there is something peculiar about the fact that a variable stands for many values, yet we exemplify or instantiate it by using an example for which only one value could hold (at a time). She eventually reduces her conflict by treating the amount of money in the wallet as, hypothetically, able to take on more than one value:

Jennifer: The amount of money in there is… *any* money in there. And after... if you like add five, if it was like... imagine if it was 50 cents, add five more and it would be 55 cents.

Fifty cents is only one of many amounts that the wallet *could have* (in principle) held.

Concluding Remarks

Elementary algebra can be viewed as a generalization of an arithmetic of numbers and quantities. Among the types of generalization we would like to draw attention to are: (1) statements about sets of numbers (e.g. "all even numbers end in an even digit") as opposed to particular numbers (2) general statements about operations and functions (e.g. "the remainder of integer division is always smaller than the divisor") as opposed to particular computations; and (3) expressions that describe relations among variables and variable quantities (e.g. "y= x +3") as opposed to relations among particular magnitudes.

This paper describes an attempt to closely document how young learners reason about problems of an algebraic or generalized nature (of the third sort above).

In the heights problem, the letters T, M, and L represent the people in the problem, their particular heights, and any and all possible values their heights might take on. Although the problem was grounded in a story about particular actors, the real story is about the relations among them, regardless of their particular values. The whole point of using multiple "stories" or scenarios in the classroom example was to draw attention to the invariant properties of the problem as the heights of individuals took on different values (varied).

The children in the study had to deal with the fact that the quantitative relations referred to particular numbers and measures on one hand (and in that sense were arithmetical), and were meant to express general properties not bound to particular values, on the other (and in this sense were algebraic). Reasoning about variable *quantities* and their interrelations would seem to provide a stage on which the drama of mathematical variables and functions can be acted out. However, using worldly situations to model mathematical ideas and relations presents students with challenging issues. On some occasions children may be inclined (as Kevin, and Melissa initially were) to instantiate variables--to assign fixed values to what were *meant* to be variable quantities--without recognizing their general character. On others, they may find it strange (as Jennifer at one point did) to use particular instances to represent variables and functions when any instance is of a constant, unvarying nature.

In future research, we hope to explore appropriate scenarios for facilitating an understanding of variables and functions beyond particular instantiations. Through close descriptions such as those provided in this paper, we hope to help the mathematics education community of researchers and practitioners uncover the true potential behind an early introduction of algebraic concepts and notations.

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