Mathematical notation to support and further reasoning

("to help me think of something").

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After carefully documenting the difficulties of algebra students (Booth, 1984; Da Rocha Falcão, 1993; Filloy & Rojano, 1989; Kieran, 1985, 1989; Laborde, 1982; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1984; Steinberg, Sleeman, & Ktorza, 1990; Vergnaud, Cortes, & Favre-Artigue, 1987), the field of mathematics education has gradually come to embrace the idea that algebra need not be postponed until adolescence (Bodanskii, 1991; Davis, 1985, 1989; Kaput, 1995; Vergnaud, 1988). Researchers have increasingly come to conclude that young children can understand mathematical concepts assumed to be fundamental to learning algebra (Brito Lima & da Rocha Falcão, 1997; Carraher, Schliemann, & Brizuela, 1999; Schifter, 1998; Schliemann, Carraher, Pendexter, & Brizuela, 1998). Many researchers and educators now believe that elementary algebraic ideas and notation should be an integral part of young students' understanding of early mathematics" To support this change in thinking and practice, the field needs research on young learners' algebraic reasoning. We take children's "algebraic reasoning" to refer to cases in which they express general properties of numbers (e.g. "whenever you divide by 2 a number that ends in an even digit, the remainder will be zero") or quantities ("regardless of how much candy John has, if he has two-thirds as much candy as Mark, then Mark has one and one-half times as much candy as John does"). Although children can spontaneously express such general properties and relations through natural language, without making use of algebraic notation, we believe that they can also express these properties and relations through written representation or notation, without having to treat conventional notation as a mere appendage to reasoning. We consider it necessary, thereby, to document how children's symbolic repertoire for expressing general properties gradually expands. From the perspective we take, children do not move suddenly from symbol-free expression to conventional written notation. Words are symbols. Diagrams are symbols. Written mathematical notation is symbolic whether or not it conforms to mathematical convention. The task before us is to document how children initially express general relations and gradually assimilate conventional algebraic notation into their expressive repertoire. It thus becomes crucial to ask ourselves how the child's algebraic reasoning itself evolves and to wonder what role, if any, the newly assimilated symbolic representations play in the course of this evolution of thinking. In line with a question posed by Kaput (1991), we wonder: "How do material notations and mental constructions interact to produce new constructions?" (p. 55).

In previous analyses (Brizuela, Schliemann, & Carraher, forthcoming) of young children's use of notations in problems requiring algebraic reasoning, we identified the gradual way in which children's notations became more and more context independent. At the beginning of the school year, the notations that children created to represent and solve algebraic problems were imbued with features peculiar to the problem at hand. For example, in representing a problem in which 17 fish had been reduced to 11 fish, children drew fish, with eyes, tails, and fins. While these notations served well the purpose of representing the problem at hand, they would probably not serve well the task of representing problems with a similar underlying arithmetical structure, such as representing how a bank balance of 17 dollars fell to 11 dollars. As the weeks went by, however, the children's notations became ever more schematic and general, focusing on the logical relationships among quantities instead of the physical properties of the quantities themselves. To further explore children's notations in early algebra, we began to consider the role that they may play in their thinking about different problems.

In his work regarding cultural tools and mathematical learning, Cobb (1995) highlights two opposing perspectives — the sociocultural and the constructivist — in the analysis of children's notations. One could argue from a sociocultural perspective that children internalize the algebraic notations used by the mathematical community. The other would argue, presumably from a constructivist perspective, that conceptual development will occur independently of the cultural tools, such as algebraic notation, that members of the learners' community make use of. Our position in this presentation is midway between these dichotomous views. That is, our task is to explore and document how the assimilation of conventional algebraic notation interacts with children's conceptual development regarding algebraic relations.

The present research was undertaken as part of an early algebra study of the authors of this paper with a classroom of 18 third grade students at a public elementary school in the Boston area during a one-year teaching experiment. The school serves a diverse multiethnic and racial community reflected well in the class composition, which included children from South America, Asia, Europe, and North America. We had undertaken the work to understand and document issues of learning and teaching in an "algebrafied" (Kaput, 1995) or "algebratized" (Davidov, 1991) arithmetical setting. Our activities in the classroom consisted of teaching a two-hour "math class" on a bi-weekly basis. The topics for the class sessions evolved from a combination of the curriculum content, the teacher's main goals for each semester, and the questions we brought to the table. During our teaching experiment, we made notations an integral part of our work with the children.

In this presentation, we will also briefly explore the connections and possible similarities between the notations developed by children and some of the landmarks in the history of mathematical notations. The similarities have to do with the types of mechanisms of thought and cognitive obstacles that can be identified in the development of mathematical notations (see Ferreiro, 1991; Ferreiro & Teberosky, 1979). We assume that the history of mathematical notations can help to shed light on developing understandings about children's notations for algebraic problems (Ferreiro, Pontecorvo, & Zucchermaglio, 1996). Charles Babbage, for example, writing in 1827 about the advantages inherent in the invention of algebraic notation, stated that,

> The quantity of meaning compressed into small space by algebraic signs, is another circumstance that facilitates the reasonings we are accustomed to carry on by their aid. The assumption of lines and figures to represent quantity and magnitude, was the method employed by the ancient geometers to present to the eye some picture by which the course of their reasonings might be traced: it was however necessary to fill up this outline by a tedious description, which in some instances even of no peculiar difficulty became nearly unintelligible, simply from its extreme length: the invention of algebra almost entirely removed this inconvenience. (in Cajori, 1929, p. 331)

In this case, we will need to explore, in the notations of the third graders' we worked with, whether their written representations of the problems help them to compress the meaning they made of the problems. A. N. Whitehead referred bluntly to this process in 1911 by stating that "by relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems" (in Cajori, 1929, p. 332). What effects might the use of written representations while solving algebraic problems have on children's reasoning processes?

Notations as tools for thinking and reflecting

The specific examples we would like to focus on in this presentation refer to Sara, a student in the third grade classroom we taught in once every two weeks. Sara exemplifies, through her actions and her words, how notations can represent not only what was done while solving a problem and what happened in the context of the problem, but also how notations can become tools for <u>thinking and reflecting about the relationships between quantities in the</u>

problem. In this way, we can begin to think about children's notations not only as tools for learners to represent their understanding and thinking about algebraic relations or as precursors of conventional algebra representation, but also as tools to further those understandings and that thinking. As Sara explained to one of us — David — during an interview with her, referring to the pie chart she had drawn to represent the fractions she was thinking about, "Well, I don't…when I draw this [the pie chart] it's just to help me think of something, so it doesn't really matter [if the pieces of the pie chart are different sizes]."

The class

At the May 28th class, the fifteenth and last class meeting with us, David and our third grade students were solving fraction problems. The first problem presented to the class to think about was:

1. Jennifer spent one-third of her money to buy ice cream. After buying the ice cream, she ended up with $6.
   How much money did she start with?
   How do you know?

   Draw a picture showing:
   Her money before buying ice cream
   The money she spent for the ice cream
   The money she had after buying the ice cream

As we had done many times before, we encouraged the students to use any kind of representation they felt comfortable with — arrows, shapes, drawings, or pie charts. The children had been introduced to the use of pie charts as notations for unit fractions by their regular classroom teacher in the week preceding this class. As the students began to think about the above problem, Jennifer proposed that the answer should be 24 — i.e., that the character in the problem must have started with 24 dollars. Explaining her solution by referring to fourths rather than thirds, she explained, "one fourth of it [the money she had] is six dollars, if you add six dollars four times it should be twenty-four." Following this, David asked for more volunteers (show videoclip):

David: Does anybody else have another analysis to give us?

Nathia: What is analysis?

David: [Sara put her hand up to participate] OK. Sara, go ahead, show us what you're understanding.

Sara: She [referring to Jennifer] said one-fourth. But it [the problem] says one-third. So you kind of draw it into parts like this, I mean like that [drawing a pie chart into thirds, with a number six written in each section of the pie].

David: How many parts do you have there?

Sara: Three.

Michael: [referring to the pie chart cut into thirds] That's a peace sign.

Sara: Six, six, and six [pointing to the three "pie pieces" in her diagram]. So six and six is twelve [pointing to two pieces of her pie chart]. And six is eighteen dollars all together [pointing to the last piece].

David: So now we have two different answers [for the total amount of money Jennifer had — 24 and 18]. Go ahead.

Sara: [reading the problem on the screen] One-third of her money on ice-cream And…but…I did that wrong [going back to her notation of the pie chart].

David: You did that wrong, why?

Sara: Because it should be like this [crossing out her first notation and then drawing a pie chart into thirds, this time with number three written in each section]. Because if she has nine dollars, this is her ice cream [pointing to one of the pieces], and then this is the six dollars that she ended up with [pointing to the two remaining pieces and referring to the six dollars mentioned in the problem]. If she spent this [pointing to one piece] for her ice cream, there's three and three [pointing to the two pieces of the pie chart that remained].

David: So now you're thinking she might have had nine dollars instead of eighteen. OK.

In this example, we see Sara trying to figure out the problem she was presented with, and using the notations she makes to help her figure her answer out. The following are the notations that Sara made that day in class, on her paper, for this problem, after having solved it in front of the whole class:

Figure 1. Sara's notations for solving the ice cream problem.

While trying to solve the problem in front of the whole class, Sara first represented her thinking about the problem; what she thought the problem was stating. She did not attempt to represent all of the actions that took place in the problem — like the buying of the ice cream and the spending of the money[1] — but instead extracted the essential information to be able to solve the problem. The first issue to sort out in this problem was that of "the thirds" — especially given that the problem had first been presented in terms of fourths by Jennifer, a very important class participant. Then, as she used the notation as a tool to reflect on the problem once again, she was able to return to problem and think about it through the lens that she had created with her notation.

In re-turning and re-flecting, she changed her thinking. Sara found that the first thirds' notation that she had made, with 6, 6, and 6, did not correspond to having two thirds of the amount being six, as had been mentioned in the problem. So using this first notation as a springboard, she only had to make a minor adjustment for her initial notation to match what was going on in the problem. Although Sara was using the type of notation for fractions preferred in

[1] Sara's writing "left" and "ice cream" could be taken to represent the different actions in the problem. These particular notations, however, were not made while she was solving the problem, and seem to express the "types" of quantities (i.e., this is the money that was left over or the money that was spent) more than the different steps of the problem.

the context of her classroom, she was still appropriating this particular notation and making use of it to figure out the problem — in a way, we may say that she was reinventing the notation.2

While using the pie to support her reasoning and problem solving processes, Sara was developing a representation that is somewhat similar to the line-segment diagrams proposed by Bodanskii (1991) and by Simon and Stimpson (1988) as a step towards students' development of algebraic representations. She used the pie and its three slices as placeholders for known and unknown amounts. The total unknown amount is a whole pie that is divided into three pieces of which two, taken together, represent 6 dollars. Although Sara did not need to write an equation to solve the problem, her approach depicts the basic structure of the relationships described in the problem and could become the foundation for an equation such as 3x-x=6 or 6+x=3x that could be used as a step towards the problem's solution.

The following episode, from later during that same class, illustrates once again how Sara was using the notation to think and reflect about the problem. The children had begun to work individually or in pairs on a second problem:

> Claudia decided to buy a book about lizards.
> Yesterday she had only one-fourth of the money she
> needs to buy the book.
> Today Claudia earned $3.00 more.
> Now she has one-half of the money she needs.
>
> How much does the book cost?
>
> Draw a picture showing:
> How much money she had yesterday
> How much money she needs
>
> Try to show where the $3.00 fits in your picture.

[2] In fact, we found that the use of the pie charts to represent unit fractions did not necessarily help the children to solve or understand the different fraction problems. We are reminded here of E. Mach’s (1906) point that “symbols which initially appear to have no meaning whatever, acquire gradually, after subjection to what might be called intellectual experimenting, a lucid and precise significance” (in Cajori, 1929, p. 330). Symbols, such as the “pie chart” don’t automatically lead to an understanding about fractions.

David noticed Sara's solution to the problem and called Analúcia over so Sara would explain her notations and her thinking (show videoclip):

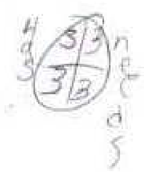

Figure 2. Sara's notation for the lizard problem.

Sara: I decided that…Claudia decided to buy a book about lizards. Yesterday she had only one fourth of the money she needs to buy the book [reading the problem]. When it said one fourth I decided I'd draw the circle with the line and the line [referring to the pie chart and the vertical and horizontal lines cutting through it — see Figure 2]. And then Claudia earned three more [continuing to read the problem]. I figured, she probably had three dollars before and then she earned three more. So I put the three down there and the three down there [pointing to Figure 2 and to the two threes on the lower half of the pie chart]. But if you go like this, that's a half and that's a half [pointing to each of the halves in the pie chart in Figure 2], so she has a half and she needs a half, so the book costs twelve dollars.

Although we were not able to follow Sara while she was solving the problem, she was able to verbalize, in considerable detail, the process she went through and how she used the notations to solve the problem. The notation she made helped her, first, to structure her thinking about the problem. When the problem stated that Claudia had one fourth, then earned three more, and finally had one half of the money she needed, she used the information in the problem to assume — correctly — that each fourth had to be the same and, therefore, each fourth of the money had to be three dollars. The notation that she developed from that inference was based on her thinking about fractions and it also helped to expand it. As was the case before, it could constitute a step towards an equation such as x+3=12/2, from which a solution could be worked out.

Later that same day, Sara, encouraged by David, made a statement about the use of different types of notations. Working on the Claudia and book about lizards problem, Sara proposed two different notations for the problem. First, she made the notation she had described for Analúcia in front of the whole class and explained it. Then, she made a second notation, saying, “I have another way that isn’t using a pie.” This is the notation that she made, representing the dollar bills that Claudia would need to buy her book:

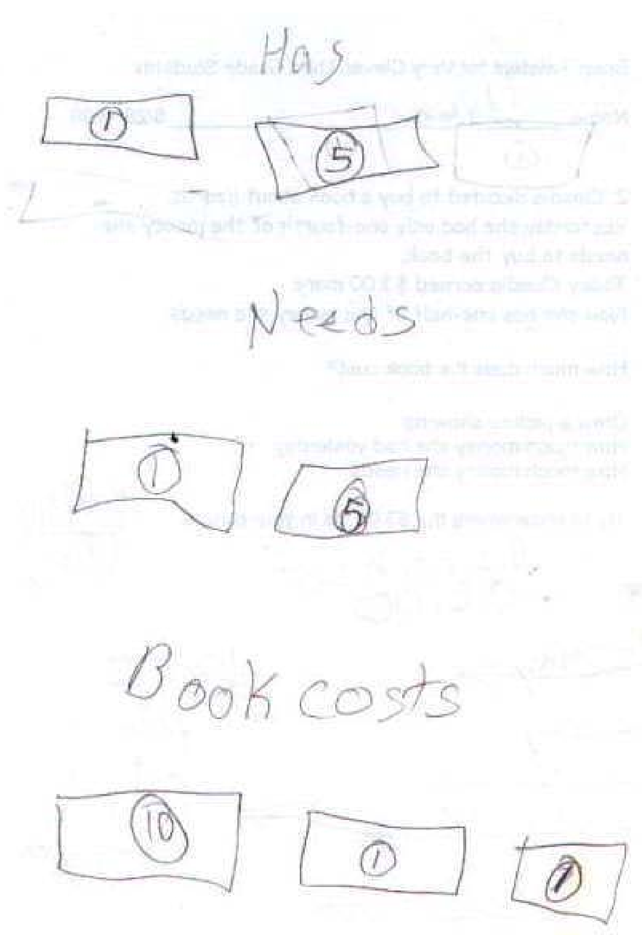

Figure 3. Sara’s second notation in front of the whole class.

When she completed this notation, David said to her:

David: You know, Sara, I think one of the…you do two different drawings. One is a good drawing if you haven't figured it out yet, and another one is a drawing that works if, only if you know, if you've already figured it out.

Sara: Yeah, if you've already figured it out this one is good [pointing to Figure 3], but if you haven't, the pie one would probably be better [see Figure 2]. If like someone already did the pie and you want to show it differently, you might want to use this one [pointing to Figure 3].

As she explained, one of these notations, namely, the pie chart, helped her to <u>think</u> about the problem, while the other, the currency representation, just showed what she did <u>after solving</u> the problem. But we might argue that while the pie chart notation helped her to <u>structure</u> her thinking, she then also used it to <u>re-structure</u> it: she reorganized the amounts (3 + 3 = 5 + 1; and 12 = 10 + 1 + 1) into relationships referred to what Claudia had, what Claudia needed, and the total cost of the book; as well as reorganizing the amounts (from $1 and $5 to $10, $1, and $1).

<u>The interview</u>

In June, after 15 classroom meetings, we had individual interviews with some of the children in the group. David interviewed Sara and Parabdeep about the following fraction problem, a follow-up of our last class in May:

Two-thirds of a fish weighs 10 pounds.
How heavy is the whole fish?

First, Sara read the problem. Immediately after reading the problem, Sara proposed a solution to the problem (show videoclip):

Sara: Twenty pounds. Because, two thirds…no, wait, fifteen pounds. Cause it would be like [drawing Figure 4] five, ten, fifteen [pointing to each one of the thirds].

<u>Figure 4.</u> Sara's notation for the fish fraction problem.

David: My goodness.

Sara: And there's ten [pointing to two of the thirds], and there's five [pointing to the third third].

David: So what is the…you drew this drawing so quickly!

[To the other student present:] Parabdeep, she didn't even give us a chance to think about it, did she? Sara, what does this mean? …. Lets read that again. Two-thirds of a fish weighs ten pounds.

Sara: I thought, I was trying to, because at first I thought it was like that [like fourths — drawing Figure 5] but then I remembered that it was this [like thirds — referring to Figure 4]. So I figured it can't be that [fourths — Figure 5], it had to be this [thirds — Figure 4].

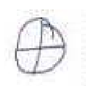

Figure 5. Sara's notation for fourths.

David: Was that in thirds over here before that you drew? [referring to Figure 5]

Sara: That was in fourths. I thought, I'm like "two" [referring to the mention of "two-thirds" in the problem], and I jumped and I thought it was four.

David: So now you did it this way? [in thirds — referring to Figure 4]

Sara: Yeah.

David: And actually you changed very quickly. And should…are these pieces the same size, or are they different sizes? [referring to the thirds in Figure 4]

Sara: Well, I don't, when I draw this it's just to help me think of something, so it doesn't really matter.

David: It doesn't really matter. But if you drew it perfectly, should you draw them the same size or different sizes? Or doesn't it matter?

Sara: The same size.

David: Oh, OK.

Sara: So, if I wanted to draw it right, it would probably look like, maybe like that. [drawing Figure 6]

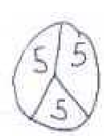

Figure 6. A “right” notation for thirds.

Here Sara is clearly stating that she is using the notation to help her think. The notation is helping her go through the problem and is helpful in allowing her to reflect about the problem. Even the notation that she has not yet made, but is still thinking about, helps her to reflect about the problem. The notation comes to be a sort of “mental image” (Piaget & Inhelder, 1971/1966) for her understanding of the problem. Objectifying and reifying this mental image (Piaget, 1976/1974; Piaget & García, 1982) she was able to reflect on it; to clarify and further her thinking about the problem.

Concluding Remarks

Like Cobb’s students using the hundreds board (Cobb, 1995), we cannot argue that Sara understood the fraction problems because she had access to the “pie chart” notation. The evidence, furthermore, is not conclusive. For example, we do not have evidence for how her thinking about fractions and her notations for them evolved. What we do have is a small snapshot into the process she went through in solving a series of problems about fractions, and hypotheses about the role that the notations might have played. In addition, we can also say that her notations helped her to reflect on the problems and to further her understanding of the problem. While the notations could at some point be a reflection of her thinking, they could also be objectified, that is, they could become an object to reflect about the problems even further (see Piaget, 1976/1974; Piaget & García, 1982). The notations Sara made helped her to think about the problems, and in that thinking — and reflecting — process, her understanding became more complex.

Sara’s initial challenge was to find a notation that would adequately help her to think about the problems at hand. As she herself explained, the pie chart helped her to think about the book of lizards problems, while the dollar bill representation (not one of the “tools” that had beenpresented to her in class) did not (see Figures 2 and 3). Similarly to Babbage’s reflection (see Cajori, 1929) about the advantages of algebraic notation, Sara’s reasonings were facilitated by the pie charts she made — diagrams that did not, however, describe in detail all that happened in the problem. By compressing meaning into her notations, Sara was able to reason about the problem, using the notation itself as a springboard and a tool for developing that reasoning.

Furthermore, we would venture to argue that Sara’s notations supported and furthered her algebraic reasoning. Her notations represented general relations among quantities. As such, they could become the foundation for the development of algebraic equations, in ways that are similar to the use of line-segment diagrams proposed by Bodanskii (1991) and by Simon and Stimpson (1988). In fact, the notations she made for the ice cream problem (see Figure 1), for the book about lizards problem (see Figure 2), and for the fish weight problem (see Figure 4) could actually stand for any other problem referring to the same quantities. The notations do not express the actions that took place in the problem, or the operations that were carried out with the quantities, but the general relations among the quantities in the problem. The notations she made helped her to change her thinking, and to re-flect on the problems presented to her.

Referring back to Kaput’s (1991) question “How do material notations and mental constructions interact to produce new constructions?” (p. 55), we could begin by saying that although Sara’s notations are not conventional algebraic notations, they do constitute an internalization of a conventional notation accepted within the context of her class, and the gradual appropriation of that notation to allow her to support and further her algebraic reasoning. That is, we take a midpoint between the dichotomous views presented at the outset of this paper by Cobb (1995)—the so-called “sociocultural” and the “constructivist.”

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