From Quantities to Ratio, Functions, and Algebraic Relations[1]

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Abstract

This study explores third graders' strategies for dealing with linear functions and constant rates of change, as they participate in activities aimed at bringing out the algebraic character of arithmetic. We discuss examples from a teaching experiment in which we gradually adapted mathematical problems involving tables so as to encourage children to describe functions with increasing clarity. Two types of building-up strategies can lead to correct answers to proportionality problems and to the design of function tables. The first type, "building up with while establishing correspondences", is widely practiced by non-schooled street sellers. The second, "building-up-blindly", is often favored by school children when asked to complete function tables. We found that although the children could correctly fill in function tables, they used building-up-blindly strategies and did not focus upon the invariant relationship between the values in the first and second columns. Several didactical maneuvers were introduced to break the students' habit of building up in the tables. Within the context of a guess-my-rule game students were finally able to focus on the functional relationship.

Everyday mathematics can constitute a solid basis for the development of school mathematics and for the meaningful learning of conventional symbolic systems. However, a student's understanding of mathematics should not be restrict to his or her former everyday experiences. The field of mathematics, although indebted to its origins in farming and commercial activities, cannot be reduced to the circumstances that gave rise to its emergence (Schlieman, Carraher, & Ceci, 1997). This perspective undermines the argument that teachers should bring out-of-school activities to the classroom or that apprenticeship training should replace teaching (Schliemann, 1995; Carraher & Schliemann, in press).

The contribution of everyday mathematics to the learning of mathematics in school is not a matter of reproducing contexts but rather a recognition that children bring to the classroom ways of understanding and dealing with mathematical problems that should be recognized as legitimate steps towards more advanced mathematical understanding. At the same time, we have to be

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aware of the differences between everyday approaches constructed as ways to reach everyday goals and the school mathematics goal of exploring multiple properties and representations of mathematical relations. This paper will explore the tension between third-graders' own ways of solving problems and attempts to expand their understanding of proportionality, ratios, and linear functions.

Functions and Rate

We explored third graders' conceptions of function and rate, specifically linear functions and constant rates of change, as they participate in activities aimed at bringing out the algebraic character of arithmetic. In principle this involves cases captured by notation such as "$y = \mathrm{m}x + \mathrm{b}$", where x and y are (independent and dependent) variables, m is a constant of proportionality, and b is the y-intercept when the function is graphed as a straight line in a Cartesian space. We all know that functions and rates involve considerably more than this. Yet mathematics educators differ widely about (1) what is critical to understand (2) how instruction should proceed and (3) how prior knowledge and experience play a role in understand function and rate.

Students begin understanding (linear) functions and (constant) rates long before they make any sense of an expression such as $y = \mathrm{m}x + \mathrm{b}$. Educators effectively teach about functions and rates long before showing such expressions to students. And certain curriculum materials embody such relations without making them explicit in algebraic notation. (A multiplication table might be thought of as an embodiment of the expression y = mx, where, x, y are integers along the margins and m corresponds to the number in the expression "times <m> table".) That mathematical concepts and ideas can be modeled, foreshadowed, dealt with intuitively, alluded to, and used as theorems in action constitutes an important fact of life in mathematics instruction as well as an important focus of research (Vergnaud, 1994). It also represents a significant departure from the view that mathematical concepts are best introduced by defining them precisely.

Studies of everyday mathematics show that quantities are crucial to the development of arithmetic concepts in the domain of additive structures. Quantities are also crucial in the development of concepts of function and rate (as well as to algebraic notation) (Schwartz, 1996). Quantities serve as forerunners of mathematical variables, essential components of functions. One commonly thinks of quantities as "what you plug in for the value of x in an equation [of the family, y = mx + b] to get out a value for y". But this view leads to a conflation of concepts of quantity, number, and measure. In the realm of mathematics education (as opposed to mathematics) quantities are *inferred qualities* of objects, *inferred properties* of collections of objects and, in most cases, mental objects that can be acted upon (Thompson, 1994). A person conceives of quantities when he or she views objects as capable of being ordered, counted, or measured along a continuum of possibilities. In this broad, psychological sense, quantities do not require numbers. This is precisely the view taken by Piaget to characterize a child's initial conception of speed as a "fastness" quality of objects (Piaget, 1946/1970). Much microcomputer-based laboratory work, such as that by Nemirovsky (1994 and in press) relies on the possibility of exploring functional relationships among quantities without requiring necessarily that students enter the world of numerical computation and additive measurement. One might also take a qualitative approach to situations where no explicit movement takes place: one can conceive of a specific door's width, for example, without measuring it or even having a particular unit of measure in mind.

By assigning numbers to quantities, either directly ("take 3 inches of rope"), or by computation or measurement, one produces measures, or measured quantities, and these representations, when given form in notation, contain information about both quantity (via a unit of measure) and number (how many units there are) (Fridman, 1990). Measures are very special mathematical objects because they offer the opportunity for children to coordinate their experience with quantities and their emerging experience with numbers and number relations. (This by no means is straightforwardly achieved.)

Function and Ratio in Non-School Settings

Let us quickly review how street sellers doing "oral mathematics" deal with quantities in situations that involve multiplicative relations (without tables) and how school children initially work with tables. Although these comparisons are not experimental, they nonetheless may serve a clarifying role.

When computing the price of a certain amount of the items they sell, street sellers start from the price of one item, usually performing successive additions of that price, as many times as the number of items to be sold (Carraher, Carraher, & Schliemann, 1985; Nunes, Schliemann, & Carraher, 1993; Schliemannn et al., 1998; Schliemannn & Magalhães, 1990; Schliemann & Carraher, 1992; Schliemann & Nunes, 1990). Vergnaud (1983, 1988) describes this strategy as a "scalar approach". The main idea is that they tend to perform repeated additions along each variable, summing money with money, items with items. If we try to understand their procedure in terms of displacements in a function table, they work down the number column and the price column, performing operations on measures of like nature.

In contrast, a functional approach presumably relies upon relationships between variables, often variables of different natures. The latter focuses on how one variable varies as a function of the other variable. Although Greeks from antiquity used internal ratios widely, external ratios and division of one measure by one of a different nature appeared in western mathematics only several hundred years ago (Freudenthal, 1983).

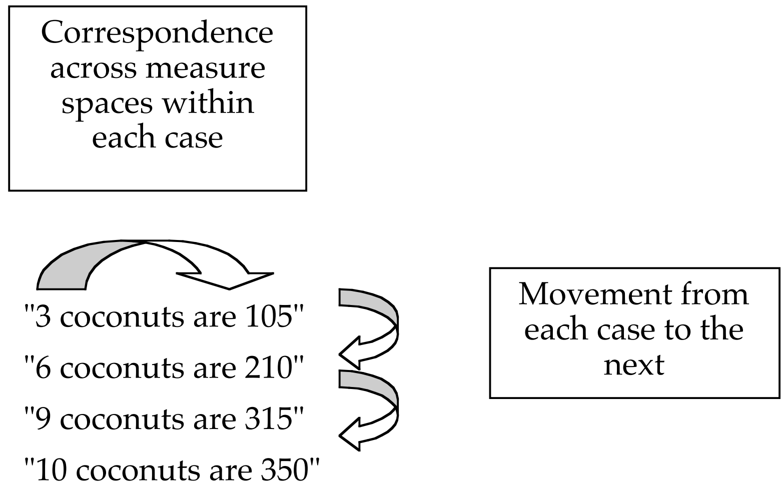
But when we look carefully at the building up strategy found in everyday mathematics research we realize that the person establishes a correspondence of values across measure spaces before proceeding to the next case. The flow of thought proceeds from one measure space to the other, row by row. This strategy is illustrated by the following solution by a coconut seller to determine the price of 10 coconuts at 35 *cruzeiros* each:

> "Three are one hundred and five, with three more, two hundred and ten (pause). There are still four. It is (pause) three hundred and fifteen (pause),

it seems it is three hundred and fifty." (Carraher, Carraher, & Schliemann, 1985).

This mental computation solution strategy can be represented by the following table and diagram:

Items	Price
3	105
6	210
9	315
10	350



Sometimes a child may build up by successive doubling; for example, a child may say, "3 cost 35, 6 cost 70, 12 cost 140" and so on. This makes use of the principle that

$$2 \times f(x) = f(2 \times x)$$

but this is appreciated in an intuitive sense and not expressed formally or generally by the child; it constitutes what Vergnaud calls a "theorem in action" as opposed to a theorem.

The street sellers' approach indeed involves a linking of a unique y-value to each value of x. It therefore captures the essential idea of a function, something easy to miss in accounts of building up and scalar solutions. We will refer to this strategy as **building up with case-by-case checks or building up while establishing correspondences**.

There are good reasons for encouraging students to explicitly formulate the relationship between measures, as opposed to merely giving a list of cases, each of which satisfies the function.

Building up can constitute a meaningful and efficient strategy to solve missing value proportionality problems, but it is limited in scope and typically does not allow for broader exploration of the relationships between the two variables. For example, if the starting amount is larger than the ending amount, street sellers fail in finding solutions (Schliemann & Carraher, 1992). Moreover, when the relationship between price and number of items (the functional relation) is numerically "easier" (Greer, 1994) than the scalar relation, school children are more inclined to focus on the functional relation while street sellers continue to use the scalar strategy, even when this choice leads to more cumbersome computations (Schliemann and Carraher, 1992).

Scalar solutions and repeated addition of the price of one item also seem to be the preferred strategy young children use in school (Kaput & West, 1994; Ricco, 1982). Children's use of scalar solutions and of unit ratios can be a good start towards understanding functions (Kaput & West, 1994). We have to be aware, however, that the goal of school instruction is to lead students to focus upon general relationships and to develop general problem solving models and notation. We also have to consider the limitations of repeated addition as a step towards understanding multiplication (Thompson & Saldanha, in press). In this intervention study we will look at some specific examples of how third-graders' emerging understanding of ratios and linear functions draws upon and at the same time departs from their previous school and out-of-school experiences with quantities and with numbers.

The intervention and its results

The data are part of a broader study with a classroom of 18 third-grade students at a public elementary school in the Boston area, serving a diverse multiethnic and racial community. The study aimed at understanding and documenting issues of learning and teaching in an "algebrafied" (Kaput, 1995) arithmetical setting (see Carraher, Schliemann, & Brizuela, 1999). Our goal was to help children build an understanding of multiplication from an algebraic point of view and as a functional relationship. To reach this goal, we designed activities that aimed at shifting the focus from scalar relations to functional relations and to general notational representation. Through a discussion of children's difficulties and successes, as they participated in these activities, we will explore some of the issues they faced in trying to move from building up approaches to a functional approach and from computations to generalizations.

We will focus on situations in which quantities *have* already been quantified (associated with a certain number of units of measure). We wish to highlight some issues students and teachers must deal in using "multiplication problems" as a springboard for students' emerging understanding of ratio, proportion, linear functions, and their growing ability to articulate these concepts in increasingly general language and algebraic notation.

We will treat algebra as a generalized arithmetic of numbers and quantities. Accordingly, we view the transition from arithmetic to algebra as a move from thinking about relations among particular numbers and measures toward relations among sets of numbers and measures, from computing numerical answers to describing relations among variables.

This approach requires providing a series of problems to students, so that they can begin to note and articulate the general patterns they see among (what we take as) variables. Tables play a crucial role in this process insofar as they allow one to systematically register diverse outcomes (one per row) and look for patterns in the results. Once the pattern has been understood, a student can fill in empty cells of the table based upon their appreciation of the underlying function.

During our intervention study, the 18 third graders in the study were intensely working on learning the multiplication tables. During the school year, we met with them once a week for a period of two hours. The first six meetings were dedicated to additive relations and were the focus of another study (Carraher, Schliemann, & Brizuela, 1999). In the seventh week, we started working on multiplicative relations. Our first challenge at this point was to design situations that would allow children to understand multiplication as a functional relationship between two quantities.

Building upon what we knew about street sellers and young children's strategies to solve price problems, we adopted this as a departure point. From our perspective, the organization of data for two related quantities in a table would presumably provide the opportunity for children to use their own scalar strategies but would also allow us to explore with them the implicit functional relationships between two variables. The sequence of tasks we designed were presented and discussed over two weekly meetings of about one hour each. The first two tasks were part of our first week meeting and the other four were part of the second.

Class 1

Task 1: Filling out function tables
On the first day, we began by asking children to fill out the table shown in Figure 1. Each child received a work sheet, but we suggested that they could work in pairs and discuss their solutions, helping each other.

Figure 1: The incomplete table

Mary had a table with the prices for boxes of Girl Scout cookies. But it rained and some numbers were wiped out. Let's help Mary fill out her table:

Boxes of cookies	Price
	$ 3.00
2	$ 6.00
3	
	$ 12.00
5	
6	
	$ 21.00
8	
9	
10	$ 30.00

Video clip 1 shows the interaction between one of the researchers (Ana) and two of the children, Yasmeen and James. These children's solutions exemplify the initial approaches observed in the classroom.

[Insert video clip 1: James and Yasmeen]

Ana: So, Yasmeen, did you see the problem? (Pointing to the first empty cell in the table) What do you think should come in here?

Yasmeen: One.
Ana: Yeah, lets write it. Now, let's go to a more difficult one.
Yasmeen: (Fills out all the empty cells in the first column)
Ana: Oh, that's clever, so now you have the numbers in the first column. So, one is three, two is six, how much should one pay for three?
James: Oh, I know this. This would be nine.
Ana: That…OK…a nine, why do you think it's a nine? Explain it to Yasmeen.
James: Because it's times like three. Three, six, nine, twelve. And it goes on and on.
Yasmeen: Yeah, because it's counting by threes.
Ana: Yeah, so, would you fill it up the next one?
James: Fifteen, eighteen.
Ana: Very good.
James: Hey! Can I take a book?
Yasmeen: (Opens up her book, checks her multiplication tables and the results of multiplying by 3 the numbers in the first column corresponding to empty cells in the second).

Most of the students in the class, like Yasmeen and James, first appeared to treat each column, items and price, as separate problems. Yasmeen discovered that she could fill out column one by counting by 1's. James "solved the column 2" problem by counting by 3's. Remarkably, their "blind-building up" approach leads to correct answers. But it does not involve them in thinking about the general relationships between price and items. The diagram below depicts the steps in their solution.

Boxes of cookies	Price
1	$ 3.00
2	$ 6.00
3	$ 9.00
4	$ 12.00
5	$ 15.00
6	$ 18.00
7	$ 21.00
8	$ 24.00
9	$ 27.00
10	$ 30.00

Yasmeen realized that she could fill out column one by counting by 1's. James has perhaps drawn Yasmeen's attention to the three-times table through his remark, "You times by three". Once Yasmeen realizes that the cookie-price table works just like multiplying by three, she consults her book to make sure her answers are in accordance with the multiplication table. A few children, like Yasmeen, related the task to the multiplication tables they were memorizing and used the latter to fill out the second column in the table.

Many students in the class, like Yasmeen and James, appear to treat each column, items and price, as separate problems. Their "blind-building up" approach leads to correct answers. But it may not be conducive to thinking about the general relationships between price and items.

Task 2: Different ways to go from one number to another
The remainder of this class was dedicated to an activity where the children had to find different ways to operate on a number in order to get to another (e.g., "How do you get from 2 to 8?" and "How do you get from 3 to 15?"). This activity constituted an attempt to have children exploring the multiple relationships between two numbers in a pair. We hoped that this would later help them to focus on determining the relationship in a function table.

The first and most popular solutions were additive solutions such as: "Add 6 to 2" or "Add 2, plus 2, plus 2." As discussions developed, children also used multiplication as alternative ways to get from one number to the other.

Class 2

Task 1: Focusing on any number (n)
The following week we first presented children with a multiplication table similar to the one they had worked with, except for an added "n" row. Our goal here was to lead children to state the general relationship depicted in the table. They were asked the questions shown in Figure 2:

Figure 2: Filling out a table and generalizing (n)

1. Last week we filled out the table below.
But now there is an extra row.
What do you think the n means?
What is the price if the number of boxes is n?
Describe what happened.

Boxes of cookies	Price
1	
2	\$6.00
3	\$9.00
4	\$12.00
	\$15.00
	\$18.00
7	\$21.00
8	
9	\$27.00
10	\$30.00
N	

Again, children easily filled in the blanks by counting by ones in the first column and counting by 3s in the second. David, the instructor for the two classes, asked them to explain how they found the number that corresponded to 4 and one child responded that he added four threes. For the same question regarding the second row, one child explained that it was three times two and another that she had added 4 to 2. For the n^{th} row one of the students, Sara, stated: "add 3 up; 11 times 3 equals 33; n probably stands for 11." Other children also considered that n was 11 and that the corresponding value in the second column was 33.

David explained that "n stands for anything." A child stated: "It could be any number." After discussion and examples, three children maintained 33 as a response in their worksheets, three left the cell blank, five adopted n+n+n or nnn as their response, and seven adopted the notation 3n or nx3. One girl wrote on her work sheet the expression nx3 followed by the equals sign: "nx3=".

Task 2: Breaking the columns' pattern
After noting the predominance of column-by-column solutions, we decided to make larger breaks in the table sequence hoping to draw children's attention to the functional relationship across columns (see Figure 3).

Figure 3: Filling out a table and generalizing to higher values and n

2. Here is another table. Can you fill in the missing values?

X	Y
1	3
2	5
3	7
4	9
5	
7	
8	
9	
10	

20	

30	

100	

N	

This table was probably more demanding since it represented a function with an additive term y = 2x +1. Let us focus on the interaction between one of the researchers (Ana) with Jessica and Sara and then with Jennifer, in video clip 2.

[Insert video clip 2: Sara, Jessica, and Jennifer]

Ana: (approaching Sara and Jessica, who had already filled out their tables up to the 10th row but did not know what to do next.) Lets see, if you time this number (pointing to 7) by something, how close does it get to fifteen?

Jessica: Twenty?

Ana: Lets see, if you say seven times something. How close do you get to fifteen? If you do seven times two, how much do you get?

Jessica: Fourteen?

Ana: To get to fifteen?

Jessica: You have to add one.

Ana: You have to add one. Lets think about that, you two. Would that same thing apply to the other numbers?

Jessica: Yeah.

Ana: Yeah? Show me for this one (pointing to 9), what happens to go from nine to nineteen?

Jessica: Nine times two.

Ana: Yeah, and then?

Jessica: Add one.

Ana: Add one. Let's see if it works for the other rows. What about this one (pointing to 3)?

Jessica: Three times two add one.

Ana: Ha-ha. So.

Jessica: OK, so, is it like, ten times two is twenty, add one, thirty! (sic).

Ana: (Approaching Jennifer) So, what did you do, Jennifer? Let me see. Yes, you did up to here (row 10). Now, lets find out what the rule is, to get to this one, here (row 20). Twenty. What's the number that should go there (in the y column for row 20)? I was telling the girls there, that if you look here, how is it that you get from three to seven? If you multiply three by something…

Jennifer: Three times…three times two?

Ana: Yeah, how much do you get?

Jennifer: Three times two is six.

Ana: OK, but you need seven.

Jennifer: Oh, I know, three times two plus one equals seven. This (pointing to row 4) is four times three (confusing times 2 for times 3, possibly because she started with the 3 x 2 example) is eight plus one equals seven, plus one equals nine.

Ana: Yeah, yeah, so?

Jennifer: So two times three plus one equals seven, I mean… (remembering that she is referring to the 20 row) two plus three… (confusing times for plus and two for three) twenty plus three is twenty-three, twenty-four, here.

Ana: Lets see. If you do two times, as you were doing here (pointing to the row above).

Jennifer: Two times three?

Ana: No. What were you doing here (pointing to 8)? Let's see. Eight times.

Jennifer: Oh, eight times two, plus one.

Ana: OK. OK. So, if you do the same here (pointing to 20).
Jennifer: Twenty times two is forty, plus one equals forty-one.
Ana: Yeah.

As we see above, the children did not spontaneously focus upon the functional relationship and needed external help to complete the table. The task was clearly difficult to them, even if taken as a simple computational routine where the same rule had to be applied to the input in order to generate an output. At the end, however, despite a few computational mistakes, they were able to apply the rule and to complete the table.

Task 3: Developing a notation for the function
The next step was to focus on a general notation for the function. David wrote the rule nx2+1 on the board and worked with the whole class, assigning different values for n and computing the result. The same was done for 3n+2. He replaced n by different numbers, including zero and 1000, and children easily computed the output.

Task 4: Finding the rule from two pairs of numbers
For the next activity, two pairs of numbers were given and children were asked to find the rule that originated them. For the first trial of this new task David wrote 3 and 6 as a first pair and 7 and 10 as a second pair. As the discussion in video clip 3 shows, the children easily found that they had to add 3 to the first number.

[Insert video clip 3: Whole class]

David: Let's work backwards, we'll start from the numbers, and you tell me what the rule is. Can you do that?
Student: Yes.
David: All right. I'm going to start, I'm not going to tell you what the rule is.
Student: You have to do it in pairs?
David: Well, yes. Hold on. I'm going to start with three.
[…]
David: Hold on, hold on. Then I'm going to go to there, to six (3 to 6). OK, attention.
Sara: Can I tell you the rule?
David: OK, and now, if I start from seven, I'm going to go to 10.
Sara: Can I tell you the rule?
David: Does anybody have the rule figured out? If I start from five… I'm going to go to…
Sara: Eight.
David: Yes, I'm going to go to eight. I think somebody knows the rule! Jennifer! What were you thinking? What's the rule?
Jennifer: Plus three?
David: Yes! If I start from n, then I have to go to what?
Students: (Inaudible).
David: Three?
Student: You have to add three.
David: I have to add three to what?
Student: To the n.
David: Yes, to the n. So how am I going to write that down?

Students: n plus three.
David: Yeah! That's the rule!
Students: Oh, we need something harder.
David: You need something harder?
Students: Yeah.

The children seem to take the rule, "n becomes n +3" as applying to all three of the cases that were presented. In this way, the n stands not necessarily for one particular instance as an unknown, but as a variable in a description of the relationship between the pairs of numbers.

The transition from understanding letters as unknowns, to understanding them as variables is notoriously difficult, even for adolescents. However, in the above activity we see third graders transitioning from the use of unknowns to the use of variables. This becomes even clearer if we look more closely (video clip 4) at the reasoning one of the students volunteered about this rule.

[Insert video clip 4: Melissa's comments]

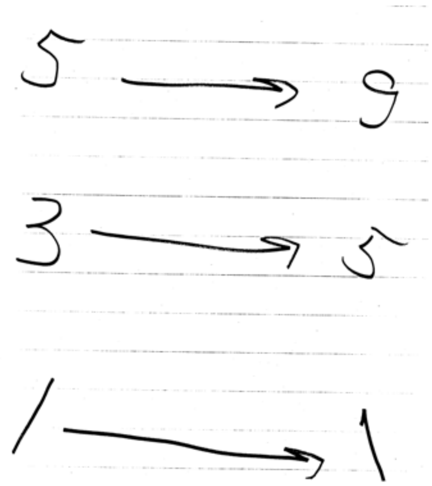
Melissa: Yeah, because you have eleven. Well. Lets just say you have ten, then you add three more and which…I mean, you have eleven, then you have three more, equal fourteen. And say if you did it with twelve, that equals fifteen.
David: Hold on. Twelve becomes fifteen. Yes. That's correct.
Melissa: And, like the higher we go, the higher the numbers get.
David: That's right. So, could you do a hundred?
Melissa: Yeah.
Students: A hundred and three.
David: That's great.
Melissa: And if you do a thousand, it's a thousand and three.
Students: A thousand and three.

Melissa's first offers the cases of 11, then 12, and then attempts to generalize: "the higher we go, the higher the numbers get". This suggests that she is referring to two *sets of numbers*, the numbers chosen ("the higher we go") as well as the numbers that emerge from applying the rule ("the higher the numbers get"). The numbers are connected one to one as ordered pairs; for each number-input there is a respective output number. But she can also mentally scan the diverse cases in an ordered fashion and think about how variations in input are related to variations in output. The numbers co-vary according a remarkably simple pattern: as input values increase, the output values increase, with the constraint that the latter are in every case precisely three units more than the former.

At the children's demand to give them "something harder", David wrote the following number pairs (see Figure 4), one by one, and asked the children to guess the rule he was using.

Video clip 5 shows the enthusiastic participation of children in this rather challenging task.

Figure 4: Input and outputs for $n \rightarrow 2n-1$



[Insert video clip 5: Whole class]

I think I better give you another example (writes 5 as an input and 9 as an output). Sara, you already want to try it?

[…]

David: OK. If I give you a three, you've got to get a five out. You think you still know? You think you know, Michael?

Michael: Yeah.

David: If I gave you an n, then what, OK…

Michael: For the first one…

David: For the first one, how do you get from five to nine?

Michael: Add four.

David: You add four. And if I add four to three?

Students: No.

David: You could've been right. Cause that's one way to get from five to nine (adding 4). However, this rule, it can't be that rule cause it didn't work for the second one (from 3 to 5). Because if I added four, this would become seven, and it became five. Let me give you another example. If I give you a one, do you know what you're gonna get from this?

James: Oh, I know!

David: James. Let's see if he's got it.

James: You have to add two?

David: You add two? So if you add two to five you get how much?

Student: Nine.

David: No, you don't get nine.

Student: Seven.

David: Actually, it's not as hard. If I give you a one, you have to get out a one.

Student: Oh, one times one equals one.

David: One times one would be one. But five times five isn't nine.

Jessica: Sara knows!

David: Sara, give us a, clarify for us.

Sara: Two times that number minus one.

David: Wow! Wow! Sara, come here, write this up here. Write it up here, if you can generalize it.

Sara: (Writes "x 2 – 1").
David: Write the n in front so we remember, n times 2 minus one. Have you guys got this figured out? Did you see what she did? So you have to use which times table, Sara? This is really something!
Students: Harder, harder.
David: Pardon me?
Students: Harder, harder.

Sara was the one who first found the rule. And it is likely that she was among a small minority of students who could infer a linear function from series of instances. However, once she gave the rule, the other children seemed to immediately recognize that it worked, and they eagerly took turns showing that they understood what was going on.

Our main point is not that students quickly and decisively mastered linear functions. We do wish to point out, however, that linear functions can begin to be explored as extensions to students' work with multiplication tables. Further, even though third grade students will not all identify the linear functions underlying data tables, they can learn significant things in the resulting discussions.

Discussion

We started our teaching intervention by using function tables relating two quantities: number of items and price. We found that although the children could correctly fill in the tables, they seemed to do so with a minimal of thought about the invariant relationship between the values in the first and second columns. Several didactical maneuvers were introduced in an attempt to break the students habit of building up in the tables.

It was only when a function-machine type game was introduced that students were finally able to break away from the building up strategies they had been using. We are not certain why this occurred. One prominent feature of the guess my rule game was that there was no discernible progression in input values to subsequent input values. This is also what happens in everyday market situations where vendors compute individual prices and, in doing so, develop an understanding about the correspondence of values across measure spaces. If this is truly an important consideration, one would suggest that we introduce function tables without ordering the rows in a systematic fashion, for example, from smaller to greater values.

It surprised us that the 3rd grade children were content to work with pure numbers and numerical relations and used this context for extracting patterns and functional relationships. Children in this school come from a multi ethnic, multi racial neighborhood, many of them from recent immigrant families. The school ranks among those with low scores in the compulsory standard achievement tests imposed by the state. Despite these drawbacks, we found that the children could focus on functional relations if the tasks they are asked to solve are conducive to examining functional relations. Surprisingly, they did not need concrete materials to support their reasoning about numerical relations and could even deal with notations of an algebraic nature. In fact, the introduction of algebraic notation helped them to move from specific computation results to generalizations about how two series of numbers are interrelated.

This short intervention opens up new possibilities for the teaching of an algebrafied arithmetic but leaves us with many other issues to be dealt with and questions to be answered such as: Were children simply following the steps proposed by the instructor, or were they developing an understanding of general functions? How do they relate the numerical relations they are able to deal with to quantities and relations included in contextualized situations or in the description of events? These are questions we hope to answer in a new study where we are following up a group of 48 children from grade 2 to 4 as they participate in activities aimed at bringing out the algebraic character of arithmetic.

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