Solving Algebra Problems Before Algebra Instruction

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Abstract

If equivalent operations are performed on the left and right terms of an equation, a new equation results. This principle allows one to produce equations with a variable isolated on one side and its value(s) on the other. It also underlies problem-solving in situations where equations are not explicitly used, but the problem calls for recognizing that two quantities are equal in value and for using that information to derive conclusions about values of unknown quantities.

The present paper focuses on how third-grade children recognize and use this logical principle in solving problems. It also looks at issues children face as they try to represent unknowns through written notation and use their written symbols to draw inferences about unknown values. The results showed that children comfortably recognized that equal additive operations upon equal quantities produce equal results (Study 1). Further, they easily produced written representations of known (numerically quantified or measured) quantities. However, the children showed considerable hesitation about producing written representations for unknown quantities (Study 2). Their hesitation seems to stem from the challenge of finding a symbol to represent a quantity without constraining or making incorrect presumptions about values it may stand for.

Introduction

If equals be added to equals, the wholes are equal.

If equals be subtracted from equals, the remainders are equal.

Euclid, “Common Notions”, *The Elements*, Book I

Much research conducted over the last two decades suggests that arithmetic instruction encourages students to think about mathematical operations as a series of givens which must be transformed through a series of one-way operations into output or "answers." The initial state is fundamentally different from the final state. For example, one begins with a certain amount of money, spends some, and then has less. Such characterizations suggest that inequivalence is the norm and that equations describe how one gets from one state (more money) to another, different state (less money). This is reflected in the common interpretation of the equal sign as meaning "gives" or "yields."

When, after years of arithmetic problem solving, students are finally introduced to algebra, the meaning of equivalence, operations, and equations undergoes a paradigm shift. Operations are meant to describe logical relations among elements (quantities or variables) instead of events or actions. In an expression such as a^2-b^2, the minus sign indicates a subtraction and yet one may be expected to factor. "Equals" no longer simply means "yields" or "gives."

Given the gulf between arithmetic and algebra, it is no surprise that research in mathematics education has consistently found that students have enormous difficulties with algebra (see, for instance, Booth, 1984; Da Rocha Falcão, 1992; Filloy & Rojano, 1989; Kieran, 1985a, 1989; Laborde, 1982; Steinberg, Sleeman & Ktorza, 1990; Resnick, Cauzinille-Marmeche, & Mathieu, 1987; Sfard & Linchevsky, 1994; Vergnaud, 1985; Vergnaud, Cortes, & Favre-Artigue, 1987; Wagner, 1981). Such difficulties seem to justify that instruction on algebra should only start when children are about 12 years old. It is only then that they are required to leave aside the direct arithmetical path and to focus, instead, on the extraction and representation of relevant mathematical relations. To help children overcome the difficulties encountered in the transition from arithmetic to algebra, researchers such as Herscovics and Kieran (1980) and Kieran (1985b) have developed teaching approaches that seek to gradually transform seventh and eighth graders’ knowledge of arithmetic, thus allowing them to build an understanding of equations. The National Council of Teachers of Mathematics (NCTM) Standards (1989) also proposes that algebra should be introduced from grades 5 to 8 as a generalization of arithmetic while “focus on its own logical framework and consistency” (p. 150) should be the goal of instruction in grades 9 to 12.

An even more radical view is emerging among researchers who believe that children should be focusing on mathematical relations while they solve mathematical problems well before grade five. Davis (1985, 1989) argues that preparation for algebra should begin in grade two or three. Vergnaud (1988) suggests that instruction on algebra or pre-algebra start at the elementary school level so that students can be better equipped to deal later with the epistemological issues involved in the transition to algebra. Schifter (1998) provides evidence of algebraic reasoning by children in elementary school classrooms. Brito Lima (1996, see also Brito Lima & da Rocha Falcão, 1997) shows that first to sixth grade Brazilian children can develop representations for algebraic problems and, with help from the interviewer, solve the problems using different solution strategies. Bodanskii (1991) discusses the tension between arithmetical versus algebraic methods for solving verbal problems and concludes that "the algebraic method is the more effective and more 'natural' way of solving problems with the aid of equations in mathematics" (p. 276). In agreement with Dieudonne's (1960, Cf. Bodanskii, 1991) views that arithmetical methods are obsolete in our times, Bodanskii proposes that, from the elementary school years, instead of teaching arithmetical problem solving, "we should teach children how to solve verbal problems by acquainting them with algebraic methods" (p. 277). This view is strongly supported by data he obtained for Russian children who received instruction on algebraic representation of verbal problems from grades one to four. Fourth graders in the experimental group used algebraic notation to solve verbal problems and performed better than their control peers throughout the school years. They also showed better results in algebra problem solving when compared to sixth and seventh graders in traditional programs of five years of arithmetic followed by algebra instruction from grade six.

There exist further epistemological and psychological reasons for focusing on mathematical relations early on in children's school instruction, thus avoiding the dichotomization between arithmetic and algebra. Epistemologically, Piaget characterized a structure as a system of relations, stressing that what are important are the relations between elements, and not the elements themselves (Piaget, 1970/1968). Thus, the number system implies a set of relations between elements, and not only elements, yields, or products. By focusing arithmetic instruction on the latter, we are ignoring one of the inherent characteristics of the number system. Psychologically, Piaget characterizes knowledge construction as "consisting in establishing relations, identifying interactions and constructing interconnections, with which the data provided by experience are organized" (in García, 1997, p. 63). Thus, an overemphasis on arithmetic instruction that ignores the relations between elements and their transformations also fails to recognize the basic ways through which we organize experiences and construct knowledge.

The contrast between children's difficulties with algebra in high school and successful attempts such as Bodanskii's to teach algebra at earlier grades suggests that it is time to seriously consider deep changes in the elementary mathematics curriculum and the possibility of having children discussing, understanding, and dealing with algebraic concepts and relations much earlier than is the norm nowadays. But such radical changes demand careful analysis of children's understanding about the logico-mathematical relations implicit in algebraic rules, of their own ways of approaching and representing algebra problems in different contexts, and of the most adequate instructional models for initiating algebra instruction. Bodanskii's (1991) study is a good start but leaves unanswered many questions concerning children's understanding of algebraic procedures, especially in what concerns the understanding of the rules for transforming equations.

The two studies we describe in this paper are preliminary investigations of how third graders understand and represent basic algebraic relations and of how they try to solve elementary algebra problems before they receive instruction on algebra. With this analysis we hope to contribute to the discussion on the appropriateness of developing algebra activities for the elementary school level. In the first study we examine third graders' understanding of one of the basic logical properties for dealing with equations, namely that, given an equality, if identical additive transformations are performed on each side, the equality remains. In the second study we examine how the same children attempt to solve verbal problems that require use of algebraic notation and algebraic rules.

Study 1: Recognizing Invariance Despite Change

One of the basic manipulation rules taught in schools for solving simple equations is expressed as "when you move one element from one side of the equation to the other side you have to change its sign." This rule is a shortcut for two additive operations, one on each side of the equation. Do students realize that adding or subtracting equal unknowns from each expression will not destroy the equality? Research suggests that this logical principle does not guide students' solutions when they try to solve equations. For instance, Steinberg, Sleeman and Ktorza (1990) found that even 13 to 15 year-old students, when asked whether two linear equations had the same solution (for example, $x + 2 = 5$ and $x + 2 - 2 = 5 - 2$), rarely justified their answers by appealing to the idea of equal operations on the left and right expressions. Instead, they worked out values for the unknown in both equations and compared their results at the end. Similar results are described by Resnick, Cauzinille-Marmeche, and Mathieu (1987) for 11 to 14 year-old French pupils. Children's typical mistakes when solving simple equations (see Kieran, 1985a, 1989) also seem to reflect a failure to understand that equivalent transformations on both sides of an equation do not alter the equality.

Why do students so often not take into account such an apparently simple logical rule? Are they unable to understand it? Or do they understand it but fail to see its relevance to the cases at hand?

In this first study, we examined how third graders understand that if we add or subtract equal amounts from each side of a given equality the terms (amounts on each side) remain equal and that, if the amounts to be added or subtracted are different, the terms will be different.

Method

Nineteen third graders from the same classroom of a public school in a Boston suburb participated in the study. Each child was individually interviewed and asked to solve eight verbal problems. For each problem, the interviewer narrated or asked the child to read a short story where two people were described as having, initially, the same amount of objects. Then equal or different quantities of objects were said to have been added or taken away from the previous equivalent amounts. The child was then asked whether or not the two people in the story still had the same amount of objects. The eight problems are shown in Table 1. The question marks around the problem structure are meant to highlight the fact that the equation reflects an issue to be solved rather than a statement of fact; indeed, sometimes the equation holds, sometimes it does not.

Insert Table 1

Problems 1 to 4 provide numerical information for all the quantities. Problems 5 to 8 state whether quantities were the same or different in amount, but do not specify numbers. For problems 1, 4, 5, and 7, the transformations on the two quantities (sides of the equation) were equivalent; for problems 2, 3, 6, and 8, different transformations were described.

For each problem, children were allowed to use whatever tools and representations they judged necessary to reach a solution and were asked to justify their answers. Paper, pencil and colored pens (magic markers) were available on the table for children to use during problem solving or during the justification phase.

Results

Children responded correctly in the great majority of cases (135 of 143 responses, or 94.4%), recognizing that equal operations to equal quantities yield equal results and that unequal operations to equal quantities yield unequal results.

Children used two main strategies to work out an answer or to justify it (see Table 2). In the first strategy, computation of values, they started from the initial amounts and then added or subtracted the amounts mentioned in the transformations for each one of the characters in the problem, comparing the results thus obtained at the end. This was children's preferred (57.9%) strategy for problems containing numerical information. In the second strategy, children focused on the transformations that took place in the story, stressing whether they were the same or different. Most of the children choosing this approach emphasized the logical necessity of their conclusions stating that, if the transformations were the same (or different), then the final quantities should be the same (or different). As many as 79.0% of the responses to the problems with no specified amounts, where no numerical computation could be performed, were of this type.

Insert Table 2

The interview transcript of the dialogue between one of the interviewers and Eliza provides some insight into how children approached the two types of problems.

To solve problems 1, 2, and 3 Eliza displayed the numerical information in each problem as two columns. She first wrote the children's names and, under each name, the respective numerical information and finally the resulting quantities. After solving problem 3, performing the written computations shown in Figure 1, Eliza explained: "I wrote the numbers to remember what to minus and what to add."

Insert Figure 1

When asked if she could solve the next problem (problem 4) without a pencil and paper, she put aside pencil and paper and started reading the problem. But after reading a few lines she gave up and said that she needed pencil and paper. The interviewer read the problem and Eliza again listed the numerical information in two columns, one for each person in the problem (see Figure 2).

Insert Figure 2

After the interviewer read the problem, Eliza started working on the computations and mistakenly concluded that, at the end, Sara had 10 and Bobby had 6 marbles. She then read the story once more and, realizing her mistake, concluded that they had equal amounts. She explained that she had thought that one of the signs that she had written was a minus sign instead of a plus sign. She justified her final answer exclusively on the basis of the arithmetical computations performed: "Both have 10 marbles at the end 'cause 8 plus 2 makes 10."

This passage shows that the presence of numbers in the problem evoked a computational response in Eliza. She was trying not to take notes but gave up and started writing the numbers in the problem, computing the results at the end. Secondly, it is significant that notation serves more than to register her thinking; it also guides her thinking. When she incorrectly read a plus sign as a minus sign, she carried out a subtraction and obtains a wrong answer.

Throughout the children's solutions to the problems with numerical information, the notations they used to keep track of the numerical information provided in the problems facilitated finding an answer. In Figures 1 and 2 we see how Eliza wrote out all of the actions that were described in the word problem. She wrote out all the sub-steps in the problem and didn't take them for granted. At times, she used the notations to "remind" herself of what had happened in the story. For example, in Figure 1 Eliza wrote "-3" at the bottom. She did not incorporate this notation into any specific computation, but seemed to have written it in order to keep track of the actions that took place in the problem. Through the notations, Eliza tells a story or re-constructs the story that was offered in words into a story represented through notation and computations. She represents the different actions that take place in the story, and then she also carries out other actions through the computations that she represents and performs. She seems comfortable enough with the notations so that she can allow herself to play around with them. In both Figure 1 and 2 she combines the computations in interesting ways, through the drawing of lines and brackets.

What happens when no numerical information is provided? While the interviewer read problem 5, Eliza wrote the names of the characters in the story and seemd to be waiting to write more information. After a while, as it became evident that no precise numerical information would be provided, she placed the pencil aside and, pausing briefly after the problem was read, she concluded that Bob and Andrew had the same number of shells:

> E: It didn't tell you how many they had so I didn't have to do any math but in the beginning it said that the boxes were equal; but it said that the bags were equal too so if both bags were gone and the boxes are equal then they'd have the same number of shells.

Eliza solved this problem mentally by analyzing the relationships between the different amounts. Instead of performing computations, she compared the equal initial state in the amount of shells each boy had, then compared the transformations, which were also equal, and concluded that this led to an equal final state.

The interviewer (I) proceeded by asking Eliza what "doing math" means:

> I: And what did you mean when you said that you "didn't have to do any math"?
>
> E: Well, it didn't really tell you how many they had, but it told you that the bags weren't there, and earlier it said that they had the same amount for the boxes.
>
> I: Um,
>
> E: So. I thought that... if the same number was for the bags and the same number was for the boxes, that if the bags weren't there then the boxes will still be even.

Eliza consulted the problem again:

> E: See, I just had to read over part of it to see...
>
> I: So, math, you mean that you just do math when there are numbers there?
>
> E: (pause) Yeah.
>
> I: And do you prefer there to be numbers, or not, or is it the same?
>
> E: Um, pretty much the same.

I (pointing to notations for problem 5): And is that why you didn't write anything for that?

E: Yeah, cause I thought there would be numbers, because usually I don't skim through it [the problem] before I read it because then I'll just have to go back again.

The above sequence illustrates a conception that may be widespread among children receiving arithmetic instruction, namely, that mathematics must involve numbers and computations as opposed to mathematics as logical relations. This conception is also illustrated by the dichotomy between arithmetic and algebra in traditional curricula.

The interviewer then read problem 6. In Figure 3 we see how Eliza sought to represent a problem that offered no numerical information that could be used in computations. Eliza started out wanting to write the parts and different steps in the story, as she had done in Figures 1 and 2 for problems 3 and 4. She wanted to keep track of the numerical information and then do something with those numbers.

Insert Figure 3

As she continued listening to the problem, she found that there was really nothing to be done with the numbers that she had represented, and she said, "It didn't tell you how many they had so I didn't have to do any math." In Figure 3 we also see how Eliza tried to use abbreviations to represent "batch" and "basket," but then found herself in trouble because both words start with the same letter. We might hypothesize that the 'b' Eliza wrote is her own version of an 'x' in a problem with unknowns. After a while, she stated that Charlie and Renee had the same amount of cookies and explained:

In the beginning each had a batch of cookies, Charlie put it in 1 basket and Rene put it in 2 but they were the same number of cookies. Charlie gave all of his basket but he still had half of the other batch and she still has half a batch. No (pause) she has more than him cause she still has the other half of the batch.

The interviewer asks her why she wrote what she wrote and she answers that:

I thought it would be harder than just figuring it our in my head if I tried it on paper cause there aren't too many numbers and it's harder to do on paper and it's easier to do in your head.

Discussion

Our results show that third graders do understand that if equivalent transformations are carried out on equal quantities the resulting quantities are equal. However, when given numerical information in problems involving equivalent transformations, they prefer to solve the problem through numerical computations. Their approach in this case is similar to that of the much older students in Steinberg, Sleeman and Ktorza's (1990) study who, when asked to judge whether two equations are equivalent, prefer to perform computations instead of reasoning about transformations. Given our findings for younger children, it does not seem that the difficulties of the older students are due to lack of logical reasoning. Instead, their computational approach may be a consequence of arithmetical training focusing mainly on computational procedures.

These results suggest that, rather then concentrating on numerical computations, we need to teach mathematics in the early grades focusing mainly on logical relations and exploring the algebraic character of arithmetic. This is consistent with what had been presented at the beginning of this paper about the epistemological underpinnings of the number system and the psychological features of knowledge construction.

But would this basic understanding of equivalence be enough to promote the development of algebraic solution strategies? What other aspects of algebraic problem solving are accessible to third graders and which ones represent obstacles to be surmounted? While they try to solve problems, would children actively use their understanding that equalities are preserved if equal transformations are carried out on both sides to derive solution strategies whereby equal unknowns appearing on both sides of an equation are cancelled out? This is what we analyzed in the second study.

Study 2: "But How Much, How Many?": The Paradox Inherent In Representing Unknowns

The issue before children in Study 1 was to determine whether two quantities remained equal after equal additive transformations were carried out on both sides of an equation. When numbers were salient they tended to carry out computations and compare the results. One might wrongly surmise from their approach that they did not appreciate the logic of equal operations to equal things. However, in the condition where numbers were not salient they appealed to this general principle. This suggests that the children may have the necessary knowledge to solve equations for unknowns.

Study 2 challenged the same children to use such knowledge in order to determine the values of unknowns. This amounts to asking children to "solve equations for one unknown," even though they were unfamiliar with algebraic notation and algebraic rules for operating on equations. Bodanskii's (1991) study showed that elementary school children can adopt and make use of a well structured notation system for solving algebra problems. However, his study left unanswered many questions concerning children's understanding of algebraic procedures, specially in what concerns their spontaneous use of notations and algebraic procedures to solve equations.

In this study we chose to focus on two linear equation problems which were also part of the study by Brito Lima (1996, see also Brito Lima & da Rocha Falcão, 1997). The first one can be represented by the equation $8 + x = 3x$. The second involves two unknowns and can be represented by $7 + y = 2 + y + x$. For each problem, canceling one unknown on each side of the equation greatly simplifies the search for a solution. But how would children proceed? What kind of notation would they spontaneously use? What kind of help is needed for this notation to be developed by the child? And, once a written representation is achieved, how do they proceed to compute a result? Would they use the canceling out strategy or some other method? These were the questions that guided our analysis of the interviews conducted with third graders while they tried to solve the two problems. It is important to emphasize that the problems were given as word problems; no equations were shown to children. We were nonetheless interested in observing the notations they would produce, spontaneously or with encouragement from the interviewer, to clarify and support their thinking. How they understood and made use of notations, even if they departed from conventions for representing equations, could inform us about how they approached the problems put before them.

Method

After solving the problems in study 1, some of the children were also asked to individually solve six new problems, in the sequence shown in Table 3. As before, the interviewer verbally presented or asked each child to read each problem and try to solve it. The interview was conducted in a flexible way, with questions and prompts that might help the child to find a path towards a solution. Children were allowed to use whatever tools and representations they judged necessary to reach a solution and were asked to justify their answers.

Insert Table 3

Problems 1 to 4 served as warm-up tasks; the two target problems of our analysis were problems 5 and 6. Not all the problems were given to all children because some of them showed that they were tired or not interested in pursuing the interview. As a result, six children were given both problem 5 and 6, nine were given problem 5 only, and four were given problem 6 only. The analysis that follows refer therefore to the responses of 15 children to problem 5 (the fish problem) and of 10 children to problem 6 (the apples problem).

Results

Of the 15 children who were given the "fish problem," two were able to independently find a solution, nine found a solution after receiving prompts from the examiner, and four children couldn't solve the problem even when help was provided. The two children who independently found a solution wrote the number 8 on paper and then proceeded to mentally solve the problem, refusing the examiner's suggestions to represent the unknown amounts in writing. When asked to explain further, one of them answered that four "would be the only logical answer." Two of the four the children who did not find a solution stated that they needed to know "how many is a few" and that the problem had no solution. The other two initiated a process of developing a written representation for the problem but appeared distracted and, although receiving help from the examiner, could not find a solution.

The initial difficulties showed by the nine children who solved the fish problem with help were related to dealing with the unknowns. Some appeared puzzled and commented that "We don't know" or that "We don't have the number." Others stated or guessed that "a few" or "some" should mean two or three. The help provided to these children initially consisted of suggestions to represent unknown quantities as some shape. Once a representation was achieved, most children would come up with a guess (usually three) for how many "a few red fish" would be. Such reaction shows that the use of a notation for an unknown left unaffected their idea that a few should be two or three. The interviewer then would suggest that they could test their guess and, when the guess led to an inequality, children were encouraged to try other numbers until an equality between the two sides of the equation was found.

Of the 10 children attempting to solve the "apples problem" five found a solution without any help, three solved the problem after receiving help, and two failed in finding an answer. Two of the children who solved the apple problem without help started by attributing value three to a few green apples. Starting with this value, one of them immediately stated that five was the value that would lead to equal amounts while the other tried different values until one that would maintain the equality was found. Two other children immediately gave a correct answer and the fifth child (Charles, see detailed description below) gave the correct answer after spontaneously drawing shapes to represent unknown amounts of apples, an approach he was led to use while trying to solve the fish problem. The two children who tried but failed to find a solution also started by attributing a value to a few green apples but were lost in the computations involved in finding out the number of yellow apples.

The apples problem entailed unknowns appearing on both sides of an equation. Such unknowns—the number of green apples—could be cancelled out or assume any value without destroying the equality. We analyzed how children dealt with them while working out a solution and on how they answered the interviewer's question about whether any number of green apples would maintain the equality. Two children spontaneously showed a clear understanding that any value could be attributed to the green apples. Two other children, as some of the children in Brito Lima's (1996) study, showed an implicit understanding that the number of green apples didn't matter but, once asked to state whether that was true, went through a few tests of different values before explicitly stating that any value would do. Although understanding that the equal amounts could assume any values without altering the equality, when directly asked to state whether the number of green apples matters, some children need tests, treating the conclusion as a matter of induction, not as a necessary logical deduction.

The following descriptions of how individual children proceeded exemplify children's typical approaches to the two problems.

One of the children (Maggie, see Figure 4), while the interviewer read the problem, took notes, organizing them along two columns. When asked to represent the quantities, she represented eight fish as eight tallies, "a few fish" by a drawing of one fish, and "three times as many fish" as a fish with three lines above it. She then guessed that Mike would have three red fish. Upon suggestion to check whether this would lead to equal amounts, she added eight plus three and compared the result to three times three, stating that three was not the right answer. She then tried five and, finally, four concluding that four was the right answer.

Insert Figure 4

Some children immediately integrated the suggestions to represent the known and unknown quantities in interesting ways as in Melinda's case, who represented "a few red fish" as a bucket of fish and "three times as many red fish" as three buckets. Two children spontaneously showed a clear understanding that any value could be attributed to the green apples, as exemplified by the following dialogue with Melanie:

While the interviewer read the problem, Melanie wrote the initials of the characters and the number of red apples each had. Without any further notation, she promptly provided the correct answer:

M: Five.

I: And how did you get five?

M: Because five plus two equals seven.

I: OK, but what about the green apples?

M: She picked the same amount.

I: Yes.

M: Kelly picked the same amount so it doesn't matter.

I: Why wouldn't it matter?

M: Because they first have, they have to start off with the same amount so they end with the same amount.

Later, the interviewer asked her to develop a representation for the unknowns, and to elaborate on the possible values for the green apples. She represented each unknown amount of green apples through the drawing of a bucket, as she had done for the representation of "a few red fish" in the previous problem (see Figure 5).

Insert Figure 5

She then further explained:

I: OK, great. Tell me how you did it.

M: Five plus two equals seven.

I: Yes.

M: And since they both have one (bucket of green apples). They have the same amount in the green bucket and at the end they have the same amount. So they get the, the umm, five plus two equals seven, and then they have the same amount in the green bucket, they have the same amount of greens, so they have, so she has five apples.

Other children needed more help as the following transcription of Charles' interview illustrates.

The interviewer started reading the "fish problem" to Charles:

I: Mike and Joe each had a water tank with fish. Mike has 8 blue fish and some number of red fish.

Charles wrote down 8 and then looked to the interviewer as if puzzled by the expression "some number of red fish" and asking for help.

I: We don't know. Rob only has red fish, but he has three times as many red fish as Mike.

Charles started to write something, but stopped and continued to listen.

I: Now, overall, Mike and Rob have the same number of fish. How many red fish does Mike have?

Charles smiled and paused. He shook his head and said:

C: I don't know.

The interviewer then proceeded by asking some questions so that he could develop a written representation for the problem:

I: How can you show 8 blue fish?

C: With an 8.

I: OK, how can you show some number of red fish?

C: I don't know.

The interviewer suggested that he use the materials on the table to draw a figure to represent the red fish:

I: OK, how about if we make a red shape on the paper?

Charles drew a red shape on the paper, next to the number 8, and colored it in (see Figure 6).

Insert Figure 6

I: OK, now, what does that mean?

C: It's... (he shrugs his shoulders).

I: It's some number, we don't know. It could be any number.

The interviewer then helped Charles represent "three times as many fish":

W (pointing to the red shape on the paper): How would you show three times that number of fish?

C: Three times three...

I: How can you show three times this little red shape here (points to the shape on the paper)?

C: But we don't have... we don't have a number for this.

I: Well, you're right, we don't have a number. How could we show three times as many red shapes?

Charles smiled nervously and the interviewer took another approach. She pointed and made a circle with her pen cap around Charles' notations and continued:

I: OK, this is Mike here, right? How about if you put Rob over here (motioning to physically separate the notations)?

C: So he has red fish.

I: How many red fish?

C: We don't know.

I: We don't know. Three times as many red fish as Mike.

C: But we don't know...

I: We don't know what the number is, but how could we draw it?

C: With a red thing?

I: How many red things? Rob has three times this many (points to the red shape drawing representing the few fish Mike has). So, what could we draw for Rob?

C: But, we don't know how many fish this is.

Given Charles's insistence that we did not know how many red fish Rob has because we do not have a number, the interviewer took a more directive approach:

I: You're right, how about if we draw three of these (points to the red shape).

Charles drew the three red shapes to represent Rob's fish. Once a written representation for all the elements in the problem was achieved, the interviewer continued:

I: Now, we know that Mike and Rob have the same number of fish altogether. So what can you tell me about this? Let's put a line here (the interviewer draws a line between the two sets of drawings).

C: It would have to be... Are those (the two sides of the drawing) the same amount of fish?

I: Yes, those are the same amount of fish on each side.

Charles paused for a moment and said, while examining his notations:

C: It would have to be... three fish here...oh,...

I: Let's start with that. If there is three fish here (pointing to the shape representing Mike's red fish), how many fish does Mike have?

C: So far we know that he has 8.

I: And, if this (pointing to the circle in Mike's side) is three fish like you said, how many does he have?

C: Eleven.

I: OK. Now, we know that Mike and Rob have the same number of fish, right? If you put three fish in each of these little circles, how many does he have?

Instead of following the interviewer's suggestion to try out number three, Charles paused for a while and then answered:

> C: So that would be 4 for each one. There are 4 in each one.
>
> I: Are there 4 in here (pointing to the shape for Mike's red fish) too?
>
> C: Yeah, there would have to be. Twelve. Twelve and twelve.
>
> I: Nice job! Way to go!

Charles smiled, looking happy about his accomplishment.

The above dialogue may suggest that too much help was being provided by the interviewer and that Charles was simply executing step by step orders, without fully understanding the problem. The "fish problem" was clearly very difficult for him, as he was convinced that he couldn't operate or represent unknown quantities. When assisted in developing and using the notation, he still had difficulties in understanding what was going on. As the interaction proceeds, at the end, aware that the two sides of his representation needed to be the same, he could test different values for the unknowns, finally finding a value satisfying the equation. His prompt reaction to the "apples problem" suggests that he was in fact learning how to use notations to solve problems. After the interviewer read the problem, Charles immediately produces the drawing in Figure 7 and promptly presented the right answer.

Insert Figure 7

Unlike Maggie and Melinda, who did not use their notations to solve the problem, Charles seemed to begin to integrate his prior strategies of trial and error and guessing with the use of notations as a meaning for reaching a solution.

Discussion

Results of this second study highlight two difficulties children must overcome to solve algebra problems. The first consists in accepting to work out a solution from unknown quantities. Children's comments stating that more information was needed to solve the problems and their frequent attempts to attribute specific values to "a few" suggest that the difficulty they encounter may result from their experiences with arithmetical problems at school, where solutions to problems are always found through operations on known quantities. The second difficulty consists in developing a notation for the unknowns. None of the children spontaneously used shapes to represent the unknowns while solving the "fish problem" and only one did so when solving the "apple problem."

Once these problems are overcome, spontaneously or after suggestions from the interviewer, they can work out a solution by trying out different values until one that satisfies the equation is found. As found in studies with adolescents (see Chazan, 1993; Fishbein & Keden, 1982; Lee & Wheeler, 1989; and Morris, 1997), they require empirical testing. This type of solution strategy, although still an arithmetic solution, may constitute a first step towards the development of children's understanding of equations and of algebraic procedures to solve them. Since they understand that equal transformations on the two sides of an equality do not destroy the equality, they may be able to later develop an algebraic solution to the equation representing the relationships in the problem. Also, their understanding that equal unknowns on the two sides of an equality can assume any value without destroying the equality might constitute a step towards the understanding and representation of variables and functions.

General Conclusions

Taken together, the results of the two studies suggest that third graders understand one of the basic principles of algebra, namely, that if equals are added or subtracted to equals, the wholes are equal. This understanding, however, remains specific to certain conditions and is only rarely explicitly used in the solution of problems involving unknowns appearing on both sides of an equality.

Although most children seem to search for arithmetical solutions attributing, from the start, values to the unknowns and testing them through trial and error, some students were able to solve the problems taking into account basic algebraic principles. For this to occur, they had to be challenged and asked to deal with unknown quantities, deriving conclusions from the relations implicit in the problems, instead of merely comparing the results of computations.

Our results suggest that third graders can develop a consistent notational system to represent the elements and the relationships in problems involving knowns and unknowns. In this process, their use of circles and shapes to represent collective bunches may constitute a meaningful transitional notation between measured quantities and unknown quantities.

We did not explore how far children would develop more advanced algebraic solution strategies to the problems and how their intuitive notations would relate to more conventional representational systems. We also did not investigate the way children intuitively deal with variables and functions, basic concepts for the understanding of more advanced algebraic problems. These should be the goals of future studies. Such explorations should help to clarify issues related to the possibility of introducing algebraic concepts and relations in the elementary school mathematics curriculum and to the choice of teaching activities that would allow the emergence of algebraic problem solving.

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Table 1: Problems Used in Study 1

Problems Given	Problem Structure (Not shown to children)
Problems With Specified Amounts	
1. Brian and Tim love to eat chocolate. One day, Brian took 10 chocolates to school and then bought 2 more at the school store. Tim brought 5 chocolates, then bought 5 more in the school store, and then got 2 more from another friend. During break time, Tim ate 2 of his chocolates and Brian also ate 2 of his chocolates. Now, do you think that after the break Tim has the same about of chocolates as Brian? Or, do you think one has more chocolates than the other?	10+2=5+5+2 10 + 2 (-2) = 5 + 5 + 2 (-2) ? [true]
2. Barbara and Joanna both had birthday parties on the same day. Barbara got 7 presents from her friends, and Joanna also got 7 presents from her friends. When each party was over, both girls had a special family time and they received more presents. Barbara received 6 more presents from her family. Joanna received 3 more presents from her family. At the end of day, do you think that Joanna has the same amount of gifts as Barbara? Or, do you think that one has more gifts than the other?	7=7 7 (+6) = 7 (+3)? [false]
3. Patricia and Daniel are neighbors playing outside. They both like oranges, so each went back to their house to get oranges. Patricia brought out 6 oranges and Daniel brought out 3. Each ran back to their house to get more oranges. Patricia brought out 4 more oranges and Daniel brought 3 more oranges. Daniel ran back a third time and returned with 4 more oranges. At this time, a friend came over. Patricia gave 6 oranges to their friend, and Daniel gave 3 oranges. Now, do you think after they shared that Patricia and Daniel had the same amount of oranges? Or, do you think that one has more oranges than the other?	6+4=3+3+4 6 + 4 (-6) = 3 + 3 + 4 (-3)? [false]
4. Bobby and Sara are playing with marbles. Bobby takes 4 marbles out of his left pocket and puts them on the ground. Bobby then takes four more marbles out of his right pocket and places them on the ground. Sara carries 8 marbles from the container and places them on the ground. After that Sara finds 2 marbles and places them on the ground. Bobby also finds 2 marbles and places them on the ground. Do you think that Bobby has the same amount of marbles as Sara? Or do you think that one has more marbles than the other?	4+4=8 4 + 4 (+2) = 8 (+2)? [true]

Problems With Unspecified Amounts

5. Bob and Andrew were collecting sea shells on the beach early in the morning. Bob put the shells he found in a big box. Andrew found the same number of shells as Bob did, but put them evenly in two small boxes. In the afternoon, they went back to the beach and Bob again found the same amount of shells as Andrew did. This time each boy put the shells they had found in a bag. The next day they went to count how many shells each one had but they could not find the bags. Do you think that Bob has the same number of shells as Andrew does? Or do you think that one of them has more shells than the other?

$x = y+y$

$x + z\,(-z) = y+y + z\,(-z)$?

[true]

6. Charlie and Renée love cookies. They each had a batch of cookies of the same amount. Charlie put all of his cookies into one basket. Renée put her cookies evenly into two baskets. Then, another batch came out of the oven. Charlie and Renée took the same amount of cookies, but this time they both put the new cookies in a bag to keep fresh to eat later. Charlie's little sister came into the kitchen where they were and said that she wanted some cookies too. Charlie gave her his basket of cookies and Renée gave her one of her baskets of cookies. Now, do you think that after they shared their cookies that Charlie had the same number of cookies as Renée? Or do you think that one had more cookies than the other?

$x+z=y+y+z$

$x + z\,(-x) = y + y + z + (-y)$?

[false]

7. Rose and Claudia collected stamps. Before Christmas Rose had the same number of stamps as Claudia did. Rose had all her stamps in one stamp book. Claudia kept her stamps in two stamp books. After Christmas they collected all the stamps from Christmas cards their families received and realized that they had each received the same number of new stamps and went to file them in their books. Do you think that now Rose has the same number of stamps as Claudia? Or do you think that one of them has more stamps than the other?

$x = y1+y2$

$x\,(+z) = y1+y2\,(+z)$?

[true]

8. One weekend Mike and Rob went fishing at the pier. On Saturday they both caught the same number of fish. Mike and Rob went back to the pier on Sunday. At the end of the day they counted how many fish each had in their buckets. They discovered that on this day Mike caught more fish than Rob. At the end of the weekend, do you think that Mike had caught the same amount of fish as Rob? Or do you think that one caught more fish than the other?

$x=x$

$y>z$

$x+y=x+z$?

[false]

Table 2 – Number of problems by type and solution strategy

Problem Type	Solution Strategies		
	Computation of values	Description of transformations	Other strategies
Numerical amounts specified (Problems 1-4)	44 (57.9 %)	24 (31.6 %)	8 (10.5 %)
Unspecified amounts (Problems 5-8)	3 (4.0 %)	60 (79.0 %)	13 (17.0 %)

Table 3: Problems of Study 2

Problems Given	Problem Structure (not shown to children)
Warm-up Problems [unknowns on one side only]	
1. Suzie and Liz spent the day selling girl scout cookies. Suzie sold 4 boxes of cookies in the morning and a few more boxes in the afternoon. Liz sold 7 boxes during the whole day. When they met at the end of the day they realized that they had sold the same number of boxes. How many boxes did Suzie sell in the afternoon?	$4 + x = 7$ $x=?$
2. David and Jenny played marbles today. At the end of the game, David had one pile of green marbles, and Jenny also had one pile of green marbles. Both piles had the same number of marbles. They counted up all the marbles and found that there were 10. How many marbles were in each pile?	$x + x = 10$ $x=?$
3. Sara and Jimmy went to the garden to pick some flowers. Jimmy picked one bunch of red flowers. Sara picked 2 bunches of red flowers. Each bunch has the same number of red flowers. When they added up all the flowers, they discovered that there were 12 flowers in all. How many flowers were in each bunch?	$3x = 12$ $x=?$
4. Mary and Joe went to different houses on Halloween. In the first house Mary received a bag of purple candies. In the second house, she received three times as many purple candies as in the first house. She then met Joe who had, overall, received 20 purple candies. Mary then counted her candies and realized that she had the same number of purple candies as Joe. How many purple candies did Mary receive in the first house?	$x + 3x = 20$ $x=?$
Target Problems [unknowns on both sides]	
5. Mike and Rob each had a water tank with fish. Mike has 8 blue fish and some red fish. Rob only has red fish; he has three times as many red fish as Mike. Overall, Mike has the same number of fish as Rob. How many red fish does Mike have?	$8 + x = 3x$ $x=?$
6. Jessica and Kelly went apple picking so that they could make some pies. Jessica picked 7 red apples and a few green apples. Kelly picked 2 red apples, the same number of green apples as did Jessica, and a few yellow apples. At the end Jessica had the same number of apples as did Kelly. How many yellow apples did Kelly pick?	$7 + y = 2 + y + x$ $x=?$

Figure 1. Eliza's notation to solve problem 3

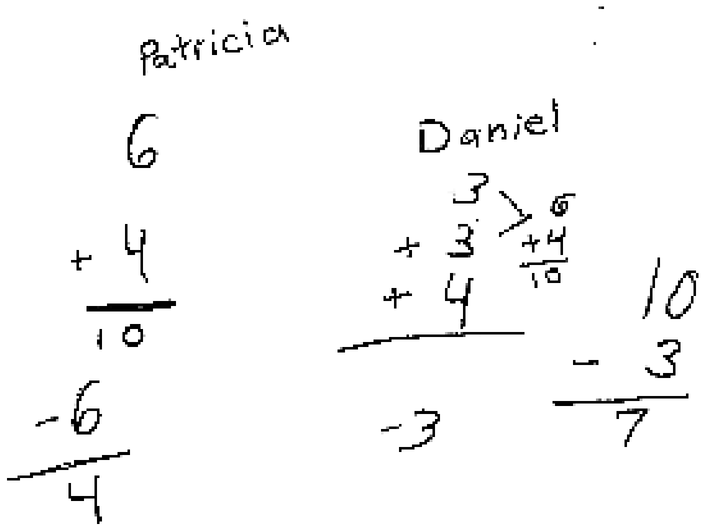

Figure 2. Eliza's notation to solve problem 4

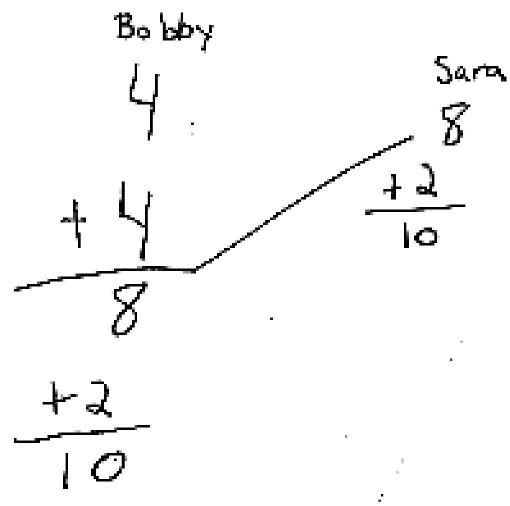

Figure 3. Eliza's notation to solve problem 6

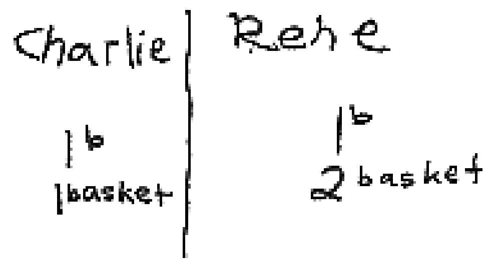

Figure 4. Maggie's notation to solve the "fish problem"

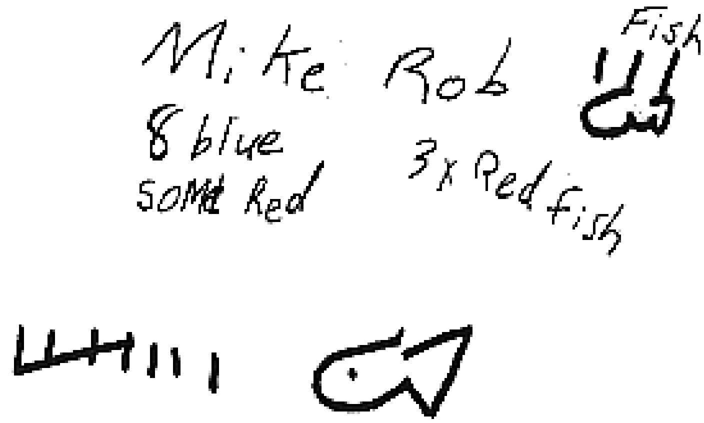

Figure 5. Melanie's notation for the "apples problem"

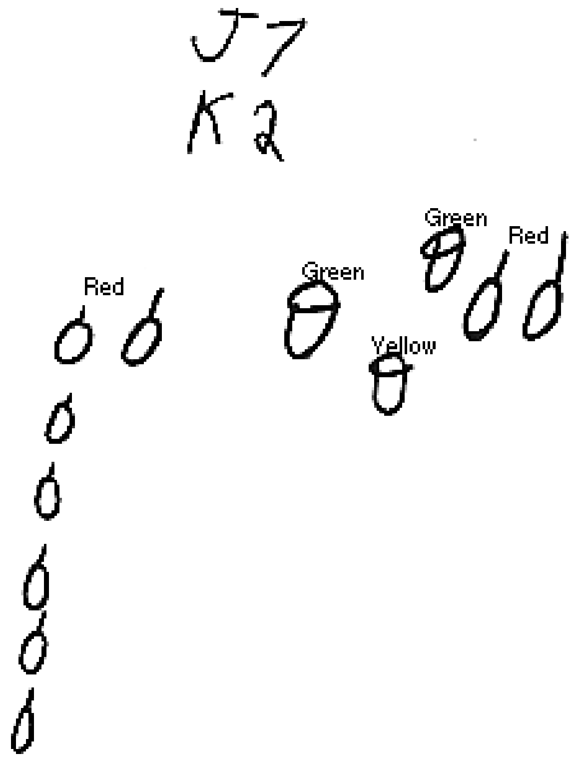

Figure 6. Charles' notation to solve the "fish problem"

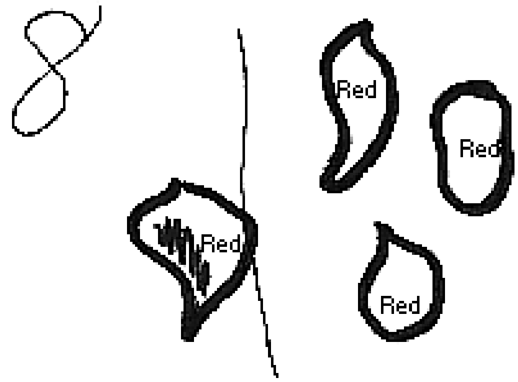

Figure 7 - Charles' notation for the apples “problem”