The Reification of Additive Differences in Early Algebra: Viva La Diférence!

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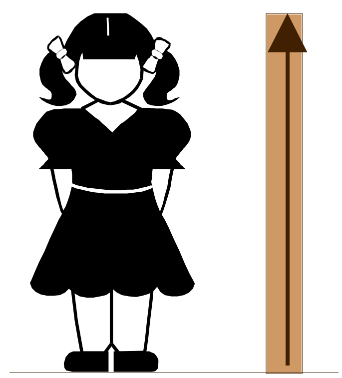
We look at the emergence of 9-year old students' concept of *additive difference*. The concept entails a tension between *process* and *object*. But even more strikingly, reifying the concept requires that children adopt analogies across diverse representational contexts. We will look at examples of students' reasoning about children's heights in contexts associated with number lines, counting, line segment diagrams, and arithmetic-algebraic notation. The examples show that *subtraction* comprises a small yet essential part of the concept of *difference*. We consider implications for research and curriculum development in early algebra and early arithmetic education.

Several years ago we began to appreciate the importance of the concept of *difference* when we designed the following problem for a class of 9-year old students (Carraher, Brizuela, & Schliemann, 2000):

> Tom is 4 inches taller than Maria.
>
> Maria is 6 inches shorter than Leslie.
>
> Draw Tom's height, Maria's height, and Leslie's height.
>
> Show what the numbers 4 and 6 refer to.



Maria   Maria's Height

The problem talks about the *differences* in heights among three characters without revealing their actual heights. This problem seemed appropriate for introducing students to additive functions. The heights could be thought to vary insofar as they could take on a set of possible values. Of course that was *our* view. The point of researching the issue was to see what sense the *students* made of such a problem. We will review our original findings and then describe what we have learned more recently through teaching experiments with other third grade students[i]. Then we will relate our findings to additive structures and early algebra.

Initially, some students interpreted the numbers as referring to the heights of the characters in the story, while others suggested adding the numbers 4 and 6 to obtain the height of the third child. Eventually the students began to talk about differences in heights and we invited three volunteers to the front of the class to represent the characters in the story. When we asked the students to point to the differences between the heights of pairs of children—that is, to show what the numbers 4 and 6 referred to—they responded in surprising ways. Typically, they would indicate the difference by placing their hand first on top of the head of one child and afterwards on the head of the other. Sometimes they would simply place their hand on the top of the taller child's head. Alternatively, they would point to where, on the taller child's body, the shorter child's top of head would reach. When we asked them whether they were referring to a single point on the taller child's body, expecting that they would sweep out the region from that point to the top of the respective child's head, they typically insisted on the single point idea. Another strategy was to place their hand on the top of one head and then move diagonally to the top of the other child's head. Each of these strategies conveyed some understanding of the difference in heights yet differed in significant ways from the convention of expressing a difference through the distance by which one quantity surpasses or falls short of the other.

We realized that our concept of difference (based on the fixed distance by which one child's head towered above or fell below that of the other along a vertical dimension) relied on conventions not yet adopted by the children. Some children seemed to represent the difference in heights through the very act of comparing (note in particular the case of sequential head-tapping). These children were more inclined to view the difference as a *process* rather than an *object*. This distinction between *process* and *object* was further highlighted when Anne Goodrow, a member of our research team, noted that children who were puzzled about where to locate a difference in heights of two students solved the problem without hesitation when it was framed by a question such as, "how much does Martha have to grow to be the same height as Paul?" (Their upward, never downward, gesticulations conveyed the process of growth.)

The present discussion evokes Sfard and Linchevski's (1994) observations on the reification of mathematical concepts. But we use *reification* in a somewhat broader sense here. We think of reification in terms of a widening of a concept across multiple contexts. We speak not merely of the increasing number of representations. An additive difference behaves and expresses its properties diversely in different representational mini-systems. In conventional representations such as those we will consider here—number lines, written algebraic notation, line segment diagrams, the subtraction and addition of sets—students must learn the conventions for expressing properties of differences and how they interrelate. The reification we refer has to do with children's travelling through the different contexts and representations.

Differences Revisited

What's the Difference?: Location vs. Distance

Two years later we took up a discussion of heights with another class of third grade students. We initially focused on enacting height differences with diverse children in the class, and in exploring the relationships between two pairs of heights through line segments and diagrams. Bárbara, who was teaching that class, asks Jennifer to show the difference between the heights of Jeffrey and Adriana (Jeffrey is a full 10 inches taller than Adriana, but no measurements have yet been taken.) Jennifer expresses the difference by pointing to Jeffrey's shoulder, which is the highest point that Adriana reaches:

Bárbara [pointing to Jeffrey's shoulder]: That is the difference?

Jennifer: No. Like, up, um…

Bárbara: Where does [the difference] start and where does it end?

Jennifer: Mm, it goes up.

Bárbara: It goes up to where?… It *starts* here. [Bárbara indicates Jeffrey's shoulder]. Let's pretend it starts here. Where would it end?

Jennifer: At his head.

Bárbara [her hands spread between Jeffrey's shoulder and the top of his head]: At his head. Could you say something like this?

Jennifer: Yeah.

Bárbara: That would be the…

Jennifer [holding her hands apart as Bárbara had]: So that would be the difference.

Jennifer faced similar difficulties to those faced by the third graders we had worked with two years before: how to represent the difference as a distance as we wanted instead of as a location, as she felt naturally inclined to do.

Reversibility in Comparisons: Taller Implies Shorter

Bárbara then introduces a ruler and with it the idea that one can measure the children's difference in heights without measuring the heights of either child. She guides Jennifer to place the ruler atop Adriana's head and to read off the number 10, which lies at the same height as the top of Jeffrey's head. The class appears comfortable calling that difference 10 inches. Jennifer explains that Adriana is 10 inches shorter than Jeffrey, and Nathan volunteers that Jeffrey is 10 inches taller than Adriana. Although this reversibility may seem obvious to us as adults, it is not necessarily obvious to many nine-year-old children. Max also understands that Adriana would have to grow 10 inches to be the same height as Jeffrey, and Risa laughs as she answers, in response to Bárbara's query, that Jeffrey would have to shrink 10 inches to be as tall as Adriana.

At this point Bárbara presents the heights problem to the children precisely as it was given two years earlier to our original class. This time the students seem to quickly understand that the numbers relate to numerical differences instead of total heights. (Although this may not be surprising, given that we had worked with the students with differences in a broad range of contexts during the previous school year and in the classes leading up to the present one.)

Expressing in Notation the Differences Between Unknown Heights

When Bárbara asks, "Tom is 4 inches taller than Maria; does it say how tall Tom is?" the class issues an emphatic "No!". The students respond likewise for the cases of Maria and Leslie. Bárbara suggests that because the characters' heights are unknown, the class could call Maria's height N. (These students were already familiar with the convention of using N to represent an unknown.)

Bárbara: Now if Maria's height was *N*, what would Tom's height be?

Students: *N* plus 4.

Bárbara: Why?

Students: Because he would be 4 inches taller.

Bárbara: Mm, hmm. And what would Leslie's height be?

Nathan and students: *N* plus 6.

Nathan: Because Leslie is 6 inches taller.

A Difference as a Subtraction

Bárbara writes the expression "Tom – Maria" on an acetate overhead and asks the class what the expression refers to. A student explains that it is the difference between Tom's and Maria's heights. Bárbara pursues the idea of subtraction.

Bárbara: So if you subtract the height of Tom's, the height of Maria from the height of Tom, what would you get?

Students: Four minus four.

Bárbara [hearing only the final 'four']: Four, right? …

Student [insisting that both fours are to be included]: **Four minus four**.

Bárbara [thinking she heard incorrectly]: Oh. Four?

Students [continuing to insist]: **…minus** four.

Bárbara [having finally written "4-4" on the overhead]: Why four minus four?

Student: Because…

Bárbara: This is the difference, the result of that four? Isn't that four inches? The height of Tom, the difference between Tom and Maria, isn't that four?

Jennifer: Mm hmm.

Bárbara: Just plain Four. Four inches.

The students seem to have been thinking that a subtraction requires a taking away of one number from another. Bárbara is actually thinking of the difference as the *result* of the subtraction and hence as a single value. This momentary misunderstanding mirrors the tension between process and object. The students are thinking in terms of the former, the teacher in terms of the latter.

Inferring the Difference of Two Differences

Shortly thereafter, Bárbara draws attention to the fact that Leslie and Tom's difference is still unknown.

Bárbara: Leslie and Tom. We have to figure out at some point what the difference between Leslie and Tom is.

Nathan: Two!

Bárbara: Why two?

Nathan: Because you…six minus four equals two.

One might think that Nathan has merely made a lucky guess. After all, given the fact that two numbers had been provided and the students were accustomed to working with addition and subtraction problems, only three distinct possibilities existed for binary operations (6-4, 4-6, and 6+4). Our past experience had shown us, however, that children were more likely to add the numbers four and six assuming they stood for total heights, to arrive at a new total heights of 10 inches for the third character involved.

Inferring the Order of the Characters in the Story

Bárbara then calls once more upon Jeffrey and Adriana to represent two of the protagonists in the story. They quickly understand that Jeffrey should represent Leslie and Adriana should represent Maria. The discussion moves to the issue of finding a student to be Tom.

Nathan [who is somewhere between Adriana and Jeffrey in height]: Can I come up?

Bárbara: Why does Tom, someone said that Tom has to be medium.

Nathan: I'm medium!

Bárbara: Why? … Well, why does Tom need to be medium, Risa?

Risa: Because he can't be taller than uh Leslie because Leslie's the tallest.

Nathan [looking for an acting role]: I'm the perfect size!

Bárbara: Leslie's the… so how come Tom can't, how come Tom can't be shorter than Maria?

Risa: 'cause Maria's the shortest, and that's saying that Tom's, like, the second shortest…

Risa's explanation may be questionable, but she is quite correct in her conclusion about the order of heights.

Making Diagrams of Unknown Heights

Bárbara then asks the students to make their own drawings of the three characters, showing what they know about the problem. In Figure 1, Jeffrey represents the heights as vertical line segments, the darkened regions of which correspond to the known differences.

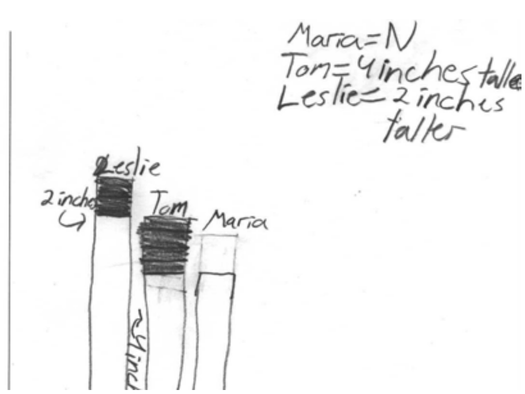


Figure 1. Jeffrey's drawing and notation for the height's problem

Ramon's (Figure 2) line segment drawing does not explicitly indicate the differences as parts of line segments. He seems to set Leslie's height at 6" and Tom's height at 4". The difference of 2" is consistent with the problem, but the numbers, 6 and 4 should correspond to differences. Here they do not. Note also that Maria's height, if indeed 6" less than Leslie's, would have to be zero inches.

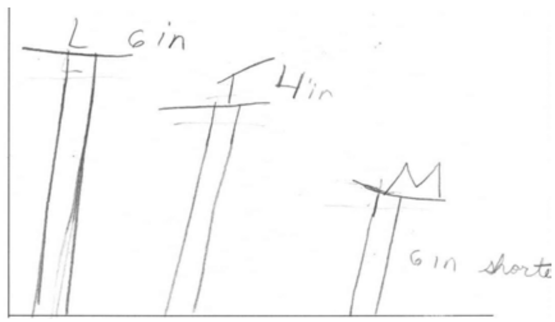


Figure 2. Ramon's drawing of the heights of the three characters

Expressing the Differences on a Variable Number Line

Jennifer notes on her own that it is possible to express the heights of the three characters in the story as positions on a variable number line, referred to in prior classes as the n-number line as opposed to the regular number line. Jennifer's diagram is shown below (see Figure 3):

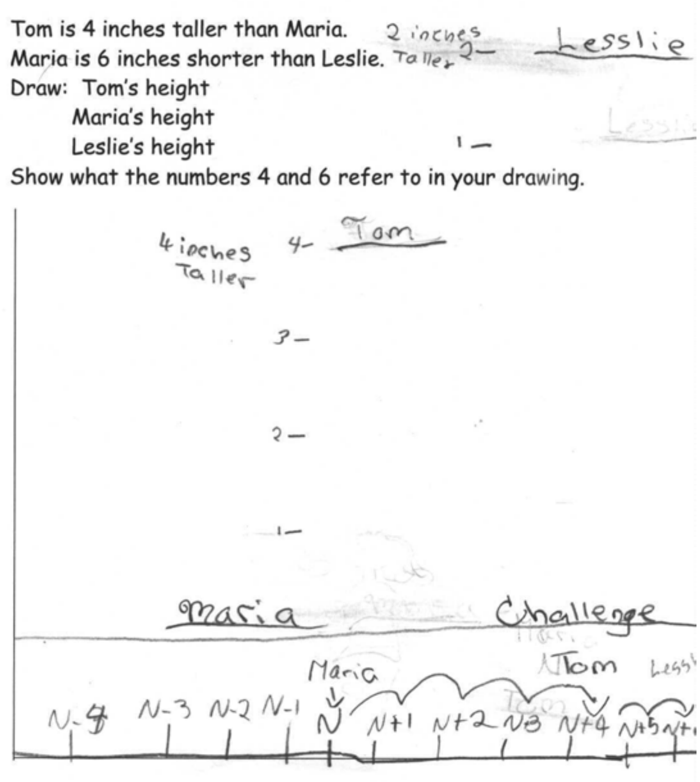


Figure 3. Jennifer's drawing (notches) showing differences but no origin. She also makes use of a variable number that forms the basis of subsequent discussion

Bárbara adopts Jennifer's number line as a basis for a full-class discussion of the relations among the heights. She further adopts Jennifer's assumption that Maria is located at N on the variable number line. See Figure 4 (middle number line) below.

Bárbara: … Now if Maria's height was *N*, what would Tom's height be?

Students: *N* plus four.

Bárbara: Why?

Students: Because he would be four inches taller.

Bárbara: Mm, hmm. And what would Leslie's height be?

Nathan and students: *N* plus six.

Nathan: Because Leslie is six inches taller.

It is remarkable that Jennifer realizes that a representational tool introduced in earlier classes would help clarify the problem at hand. It is equally impressive that the remaining students appear comfortable with the idea and easily infer the values of Tom and Leslie from Maria's value.

Bárbara wonders to herself whether the students realize that the decision to call Maria's height *N* was arbitrary. So she asks the students to assume instead that Leslie's height was *N*. The students immediately argue that Tom would be at *N*-2 on the number line and Maria would be at *N*-6. See Figure 4 (bottom number line) below.

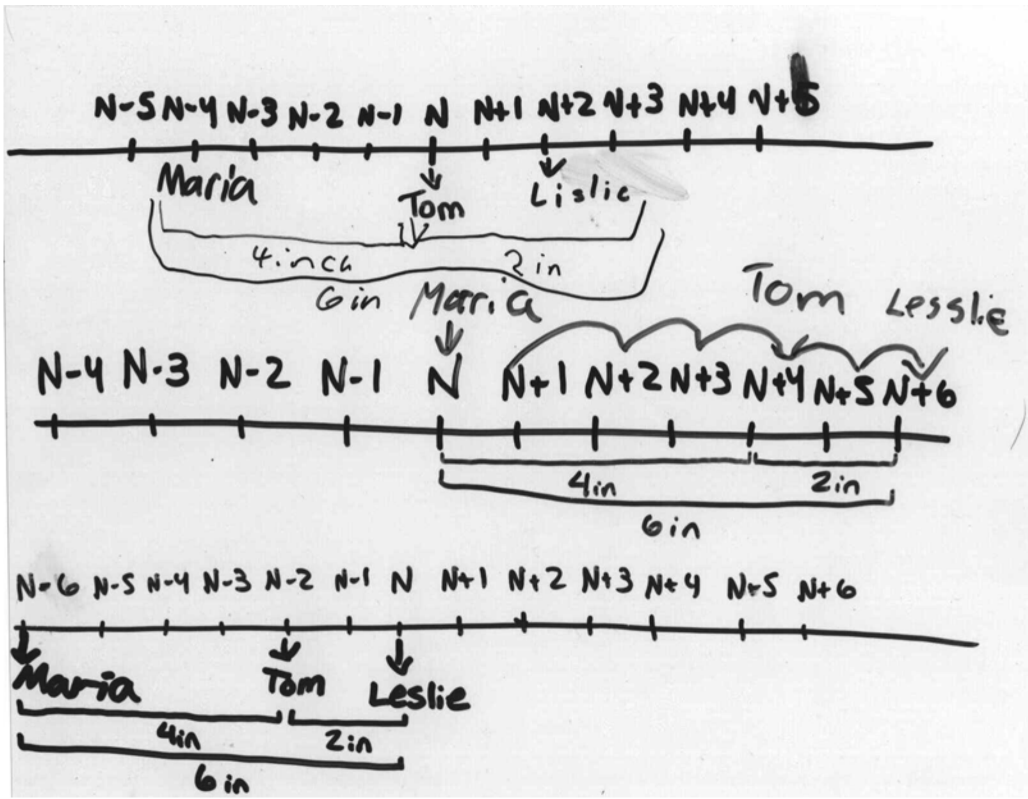


Figure 4. Three variable number line representations (on overhead) used by students and teacher to discuss the cases where [middle] Maria is attributed the height of *N*; [bottom] Leslie is assigned the height of *N*; and [top] Tom is assigned a height of *N*

Numbers as Fence Posts vs. Numbers as Intervals or Distances

Bárbara then asks the children to assume that Tom had been assigned the value of *N*. Max goes to the front of the class and places Leslie at *N*+3. See Figure 4 (top number line) below; there is an erasure under *N*+3 where Max had first incorrectly put Leslie's name. Why does he do this, even when questioned by Bárbara? Max realizes that the difference between Tom and Leslie is two, but nonetheless places her three units to the right of Tom. This is an example of the "fence post" issue. Students are well accustomed to the idea that a number refers to the count of elements in a set, that is, a set's cardinality. However, the issue before children often is: what should I count? On a number line two sorts of elements suggest themselves. One can count the number of "fence posts" or notches: each numeral is located at a notch along the number line. In Max's case he seems to have counted the number of numerals lying between *N* and *N*+3, the delimiters. Mathematical convention dictates that one count the number of *unit intervals* separating the delimiters. In the present case, one counts the two intervals, [*N*, *N*+1] and [N+1, N+2]. This may seem like a minor issue, but if a student is thinking of the number of integral points on the line rather than the number of unit-distances or intervals, misunderstandings are likely to arise in a wide variety of situations.

But if we focus too much on such momentary adjustments that students like Max may be required to make, we may fail to see the larger picture; namely, that by the end of the lesson the students are relating the given numerical differences to a number of symbolic representations: algebraic notation, line segment diagrams, number lines (including variable number lines), subtraction, counting, and natural language descriptions. The concept of additive difference does not reside in any one of these representations. And we cannot observe the students' concepts directly. But the fluidity with which students move from one representational context to another assures us that their understanding of *additive difference* is robust and flexible.

Concluding Remarks

An *additive difference* is a rich mathematical concept that manifests itself in a wide variety of contexts and through a diversity of representational forms. Although it is related to subtraction, it is not reducible to subtraction. Although it is related to displacements on a number line, it does not reduce to the idea of a number line displacement. An additive difference plays a key role in the emergence of early algebraic understanding in that it is central to the concepts of addition and subtraction and their conceptualisation as functions. But it does not end there. Additive structures lie at the heart of many advanced ideas in mathematics, including fractions, measurement, and even statistical concepts such as the analysis of variance. It is important to begin to nurture their development from very early mathematics education. Along the way, the term, *difference,* may begin to serve an integrative function, a linguistic handle that unites otherwise disparate situations. (Learning the concept entails far more than learning to use the term; nonetheless the linguistic representations of the concept are very important symbolic vehicles that should not be underestimated in mathematics education.)

Students may initially treat differences as the results of processes or actions (of growing, moving etc.) that change as an entity moves from one state to another. This is a legitimate way of thinking about differences, which never needs to be abandoned. But students also need to view differences as bona fide quantities that can be represented much in the same way as states can. This process-object tension is useful for us to gain an initial appreciation of the diversity of meanings the concept may take on. However, in order to develop well thought-out theories of learning and curriculum development we need to go far beyond this general characterization. We need to look at the particular ways in which notation can help students represent and work through specific mathematical issues (Brizuela, Carraher, & Schliemann, 2000). When students move from one mini-representation, such as from a set approach to a number line representation, particular kinds of issues arise that cannot be anticipated by a general theory of reification. We used the fence post issue as a case in point. We could highlight others, such as learning to distinguish between first and second-order differences (differences of differences) or learning to move between instantiation and generalization (for example, using N both to represent a particular value and the set of all possible values).

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