Hands On!

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Algebra in the early grades

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Consider the following problem:

Mary and John each have a piggy bank.

On **Sunday** they both had the same amount in their piggy banks.

On **Monday**, their grandmother comes to visit and gives $3 to each of them.

On **Tuesday**, they go together to the bookstore. Mary spends $3 on a book. John spends $5 on a calendar with pictures of dogs on it.

On **Wednesday**, John washes his neighbor's car and makes $4. Mary also made $4 babysitting. They run to put their money in their piggy banks.

Show how much money Mary and John have on each day. Compare their amounts for each day. Show how much money they end with.

This is no ordinary arithmetic problem. It doesn't say how much money John and Mary had on Sunday. And you cannot know how much they have on any day. So what's the point?

If you think about it, you can compare Mary's and John's amounts on each day, determine who ends up with more money, and learn whether each child ends with more or less money than they started with. To do so, you need to examine the relations among the amounts and work with unknown values, drawing conclusions that are true for whatever amounts the children began with. This is a problem of "algebraic arithmetic."

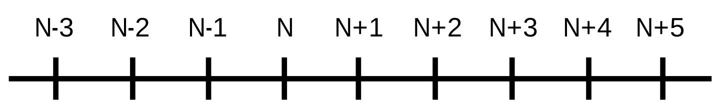
Although K–12 mathematics curricula have long been built on the assumption that arithmetic and algebra are distinct areas of mathematics, there are reasons for treating them as intertwined and overlapping. Traditional arithmetic requires students to perform calculations on particular numbers, but students also need to move from thinking merely about individual values to looking at sets of values. If we tell students the initial amounts for the "piggy bank" problem, they can simply calculate the answer. Without defining the amount, students must represent and operate on unknown values.

In the Early Algebra, Early Arithmetic project at TERC, we have been investigating how young students think about and represent functions and unknowns, using both their own and conventional symbols. When presented with tasks similar to the "piggy bank" problem, third grade students participating in the project have been doing some remarkable mathematics.

The following is an account of how one class and their teacher, Bárbara, worked through the problem.[1]

Representing An Unknown Amount

The students were first given the problem in its entirety, so that they could understand that it consisted of a number of parts. Then they received a problem sheet with just Sunday's information. The sheet also had the following variable number line (or N-number line):



The students worked alone or in pairs, trying to represent on paper what was described in the problem.

Sunday. After Kimberley reads the Sunday part for the whole class, Bárbara asks whether they know how much money Mary and John have. In unison the children exclaim "no" and do not appear to be bothered by that. The children state that the amount is "any number" and "anything" and a few suggest it is "N." Talik offers, "N, it's for anything."

Bárbara asks how they should represent the first step in the problem. Filipe says, "You could make some money in them [the piggy banks], but it has to be the same amount." When Bárbara reminds him that he doesn't know what the amount is, he suggests writing N. Bárbara tells the students that they can use the N-number line on their problem sheet. She also draws a copy of it on the board.

Jennifer uses N to represent the initial amount in each bank. She draws two piggy banks, labeling one for Mary, the other for John, and writes next to them a large N along with the statement "Don't know?" David (from the project research team) points to "N" on her handout and asks:

[1]The students attend a public elementary school in a multi-cultural, working-class community.

David: Why did you write that down?

Jennifer: Because you don't know. You don't know how much amount they have.

David: […] What does that mean to you?

Jennifer: N means any number.

David: Do they each have N, or do they have N together?

Jennifer: (No response.)

David: How much does Mary have?

Jennifer: N.

David: And how about John?

Jennifer: N.

David: Is that the same N or do they have different Ns?

Jennifer: They're the same, because it said on Sunday that they had the same amount of money.

David: And so, if we say that John has N, is it that they have, like, ten dollars each?

Jennifer: No.

David: Why not?

Jennifer: Because we don't know how much they have.

The children themselves proposed using N to represent an unknown quantity. The researchers had introduced the convention before in other contexts but now it was making its way into their own repertoire of representational tools. Several children appeared to be comfortable with the notation for an unknown as well as with the idea that they could work with quantities that might remain unknown. Some started by attributing a particular value to the unknown amounts in the piggy banks but, as they discussed what they were doing, most of them seemed to accept that this was only a guess. Their written work shows that by the end of the class 13 of the 16 children adopted N to represent how much money Mary and John began with. One child chose to represent the unknown quantities with question marks and only two children persisted using an initial specific amount in their worksheets.

Talking About Changes in Unknown Amounts

Monday. The children infer that Mary and John would continue having the same amount of money as each other, and that they both had $3 more than the day before. As Talik explains, "Before they had the same amount of money, plus three, [now] they both had three more, so it's the same amount."

Bárbara asks the children to propose a way to show the amounts on Monday. Most of the children use N in their depictions. Nathan proposes that on Monday they would each have N plus 3 "because we don't know how much money they had on Sunday, and they got plus …and they got three more dollars on Monday."

Jeffrey offers a drawing (Figure 1) as an explanation. Three units are drawn on top of each quantity, N, of unspecified amount. Some students use a question mark in their representations. Filipe represents the amount of money on Monday as "?+3." Bárbara comments on Filipe's use of a question mark. He and other children acknowledge that N is another way to show the question mark.

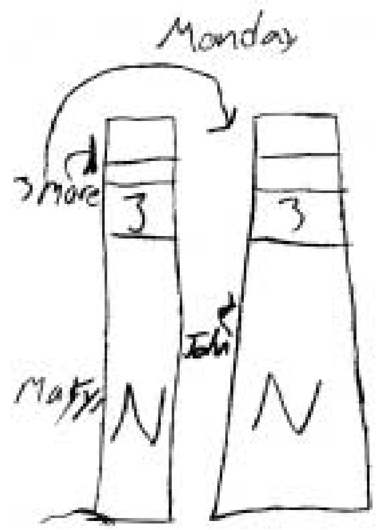


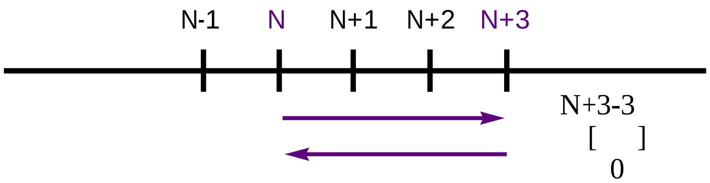
Figure 1. Jeffrey's representation of N+3

Tuesday. Mary and John have begun to spend money and that makes some of the students uncomfortable. They want to make sure that both have enough money in their piggy banks to cover what they spend. One student supposes that they probably have ten dollars, most assume that there is at least $5 in the piggy banks by the end of Monday, otherwise John could not have bought a $5 calendar. (They seem uncomfortable with him spending money he doesn't have.)

Bárbara recalls for the class what happened on Sunday and Monday. The children agree that on Monday Mary and John had the same amounts. In response to Bárbara's question about the amounts on Tuesday, the children agree that Mary and John will have different amounts of money because John spent more money than Mary.

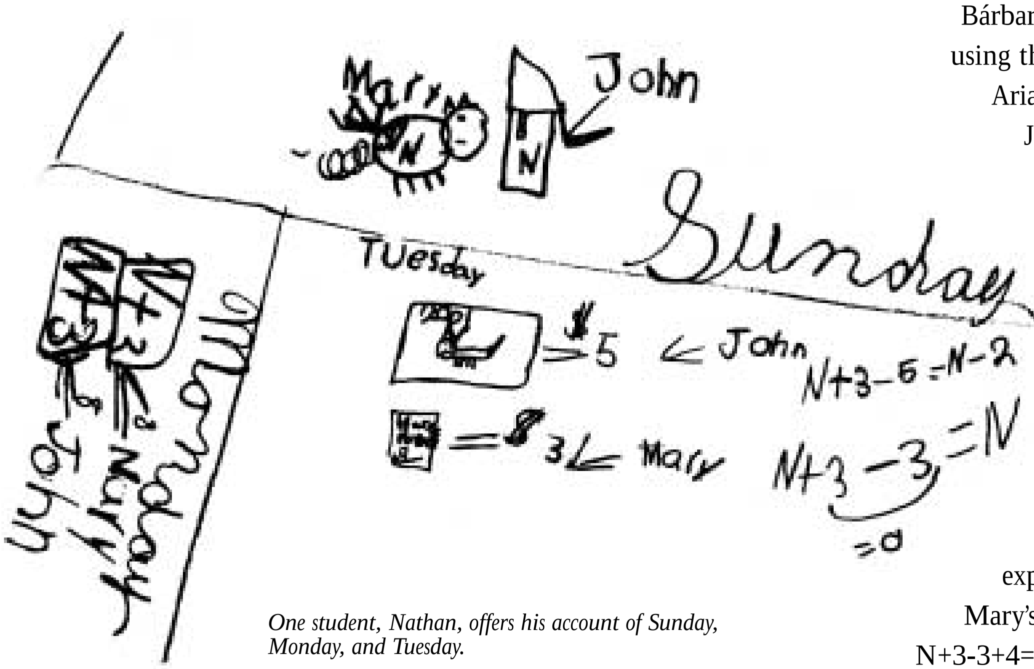
Jennifer describes what happened from Sunday to Tuesday and concludes that on Tuesday Mary ends up with the same amount of money that she had on Sunday, "because she spends her $3." Bárbara encourages the children to use the N-number line to represent the transactions from Sunday to Tuesday.

Continuing the dialogue with students, Bárbara draws green arrows going from N to N+3 and then back to N again. She uses notation as well and writes N+3-3. She puts a bracket under +3-3 and a zero below it, commenting that +3-3 is the same as zero, and extends the notation to N+3-3=N+0=N.



Jennifer then explains how the $3 dollars spent negates the $3 given by the grandmother, "Because you added 3, right? And then she took, she spent those 3 and she has the number she started with."

Using the N-number line Bárbara leads the students through John's transactions, drawing arrows from N to N+3, then N-2, for each step of her drawing. While sketching each arrow, she repeatedly draws upon the students' comments to arrive at the notation, N+3-5. Some children suggest that this is equal to "N minus 2." Bárbara continues, writing N+3-5=N-2. She asks Jennifer to use the number line in the front of the class to point to the difference between John's and Mary's amounts on Tuesday. Jennifer first points ambiguously to a position between N-2 and N-1. When Bárbara asks her to show exactly where the difference starts and ends, Jennifer correctly points to N-2 and to N as the endpoints.



One student, Nathan, offers his account of Sunday, Monday, and Tuesday.

David (the researcher) asks Jennifer how much John would have to receive to have the amount he had on Sunday. She answers that we would have to give two dollars to John. Using the number line, she explains that if he is at N-2 and we add 2, we get back to N. Bárbara represents what Jennifer has said as: N-2+2=N. Eagerly grabbing the marker, Jennifer brackets the sub-expression, "-2+2," and writes a zero under it. Bárbara asks why it equals zero. Together with Jennifer, she goes through the steps corresponding to N-2+2 on the number line and lands at N. Talik shows how this works if N were 150. Bárbara uses Talik's example to demonstrate how, given a specified amount like 150, you always return to the point of departure on the number line.

Wednesday. Bárbara asks whether Mary and John will end up with the same amount on Wednesday. James says "No." Arianna explains that Mary will have N+4 and John will have N+2.

Bárbara asks Arianna to tell the story using the N-number line on the board. Arianna represents the changes for John and for Mary. Bárbara then writes out the notations, N+4=N+4, then N-2+4=N+2. Talik explains this by saying that if you take 2 from the 4, it will equal 2. To clarify where the 2 comes from, Bárbara represents the following operations on a regular number line: -2+4=2.

Bárbara asks if anyone can explain the equation referring to Mary's situation, namely, N+3-3+4=N+4. Talik again volunteers and crosses out the +3-3 saying that it isn't needed anymore. This is a significant moment because no one has ever introduced the procedure of striking out the sum of a number and its additive inverse (although they had used brackets to simplify sums). It may well represent the meaningful emergence of a syntactical rule.

Bárbara brackets the numbers and shows that +3-3 yields zero. She proposes to write out the "long" equation for John, N+3-5+4=N+2. The students help her to go through each step in the story and build the equation from scratch. But they do not get the result, N+2, immediately. When the variable number line comes into the picture they see that the result is N+2.

When Bárbara asks Jennifer to show how the equation can be simplified and Jennifer hesitates, Bárbara points out that this problem regarding John's amount is harder than the one regarding Mary. Bárbara asks her to start out with +3-5; Jennifer says -2. Then they bracket the second part at -2+4, and Jennifer, counting on her fingers, says it is +2 and writes it out. Talik explains, "N is anything, plus 3 minus 5 is minus 2; N minus 2 plus 4, equals (counting on his fingers) N plus 2.

Talik then tries to group the numbers differently, adding 3 and 4 and then taking away 5. Bárbara points out how the numbers could be grouped a different way and shows that +3+4 yields +7. When she subtracts 5, she ends up at +2, the same place suggested by Jennifer.

Thursday. The students are given the final part of the problem and learn that Mary ends with $9 in her piggy bank. Several students respond that N has to be 5. Bárbara asks the children, "How much does John have in his piggy bank?" Some say incorrectly that he has two more; other children say that he has 7. Some of the students figure this out from adding 5+2, others from the fact that John was known to have 2 less than Mary, since N+2 is two less than N+4.

Bárbara ends by filling out a data table that includes the names of Mary and John and the different days of the week with the children's suggestions for how much money each one had on each of the different days. Some students suggest using expressions containing N and others suggest expressions containing the now known value, 5.

Some Reflections

The lesson described above is typical in several ways of the 12 lessons carried out with the students in three grade 3 classrooms. The students' responses were diverse, with some relying more than others on instantiating unknowns to particular values. Over time, however, in each lesson and across the lessons the students increasingly came to use algebraic notations and number line representations as a natural means of describing the events of stories.

Our experience has convinced us that children as young as eight and nine years of age can learn to comfortably use letters to represent unknown values, and can operate on representations involving letters and numbers without having to instantiate them. To conclude that the sequence of operations "N+3-5+4" is equal to N+2, and to explain, as many children did, that N plus 2 must equal two more than what John started out with, whatever that value might be, is a significant achievement—one that many people would think young children incapable of understanding. Yet we found such cases to be frequent and not confined to any particular kind of problem context. It would be a mistake to dismiss such advances as mere concrete solutions, unworthy of the label "algebraic." Children were able to operate on unknown values and draw inferences about these operations while fully realizing that they did not know the values of the unknowns.

By arguing that children can learn algebraic concepts early we are not denying the developmental nature of these concepts, much less asserting that any mathematical concept can be learned at any time. Algebraic understanding will evolve slowly over the course of many years. But we need not await adolescence to intervene in its evolution.

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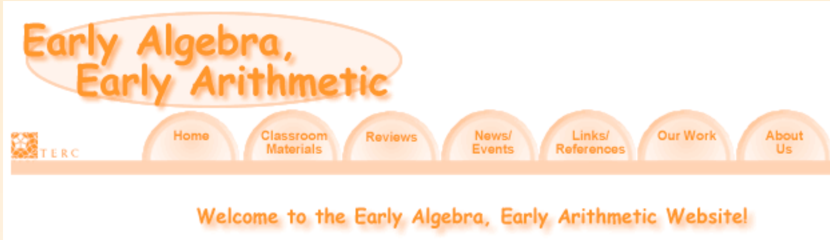

What is Early Algebra?

Early Algebra is an approach to early mathematics teaching and learning. It includes many topics in arithmetic, such as the four operations, but it does so in novel ways. Early algebra does not aim to increase the amount of mathematics students must learn. Rather, it is about teaching time-honored topics of early mathematics in deeper, more challenging ways.

Early Algebra is also an area of research. Several mathematics educators have suggested that algebra should enter the early mathematics curriculum (and they have initiated systematic studies. (The Early Algebra, Early Arithmetic web site has an extensive bibliography of research in this emerging field.)

Still, much remains to be done. We view algebraic arithmetic as an exciting proposition, but one for which the ramifications can be known only if a significant number of people undertake systematic teaching experiments and research.

To learn more visit the Early Algebra, Early Arithmetic web site **www.terc.edu/earlyalgebra/.**