Functions and Graphs in Third Grade[1]

Analúcia D. Schliemann
Tufts University
aschliem@tufts.edu

Anne Goodrow
Boston Public Schools
agoodrow@gis.net

Susanna Lara-Roth
Tufts University
slararot@worldnet.att.net

Teaching about multiplicative functions is traditionally postponed until the middle or high school years. It seems, however, that children are able to deal with functional relationships at an earlier age. In this paper we analyse how second graders complete function tables and how instructional activities involving ratios and graphs may encourage third graders to focus on functional relationships.

Data by Piaget, Grize, Szeminska, & Bang (1968/1977) show that nine-year olds are able to quantify functions involving direct proportions. If children can deal with functional relationships this early, work on functions should not be postponed until middle or high school. How could this be done? How can we build upon children's initial understandings about quantities and number relations as a foundation for learning about multiplicative functions in elementary school? What changes in their initial approaches to tables could take place as they participate in instructional activities that focus on functional relationships, algebraic notation, graphs, and variables?

In our earlier work (Nunes, Schliemann, & Carraher, 1993) we found that street sellers compute the price of a certain amount of items by performing successive additions of the price of one item, as many times as the number of items to be sold. The following solution by a coconut vendor to determine the price of 10 coconuts at 35 *cruzeiros* each exemplifies this:

> "Three will be one hundred and five; with three more, that will be two hundred and ten. [Pause]. I need four more. That is… [Pause] three hundred and fifteen… I think it is three hundred and fifty." (Nunes, Schliemann, & Carraher, 1993, p. 19).

The street sellers perform operations on measures of like nature, summing money with money, items with items, thus using a scalar approach (Vergnaud, 1983). In contrast, a functional approach relies upon relationships between variables and on how one variable changes as a function of the other. While work with scalar solutions can constitute a meaningful first step towards understanding number or quantities (Kaput & West, 1994), a focus on scalar solutions does not allow for broader exploration of the relationships between two variables (Schliemann & Carraher, 1992; Carraher & Schliemann, in press). Schools should therefore provide children with opportunities to explore functional relationships.

In the two studies reported here we look at how, by the end of second grade, children complete multiplicative function tables. After that we provide examples of how, in third grade, their understanding of functional relations may be expanded. The data come from a broader study on bringing out the algebraic character of arithmetic.

---

[1] Symposium Paper, NCTM 2001 Research Presession. Orlando, FL. Support for this study was provided by NSF Grants #9722732 and #9909591. Opinions expressed are those of the authors and not necessarily those of the Foundation. Special thanks to Fred O'Meara, Thelma Davis, the teachers and children in our project and to D. Carraher, B. Brizuela, D. Earnest, and J. Karacz for their cooperation.

Study 1: Second Graders Working with Tables

We interviewed 28 second-graders, in the middle and at the end of the school year, at a public elementary school in the Boston area serving a diverse multiethnic community. In each occasion we asked them to answer questions and complete the tables shown in Figures 1 and 2.

Problem 1: On Prices in a 1 to 3 Ratio

How much is one box? In the first interview (mid-school year) 19 (68%) children found that one box cost three dollars. Some promptly gave the right answer and explained it by skip counting or by successive additions of three. Others guessed a price and tested it until they found a number that added four times would lead to 12. Some children counted tokens one by one or three by three and others inferred the price of one box from the price of two boxes on the table. At the end of the school year (second interview) 22 (78%) children correctly determined the price of one box. Of these, only three needed help from the examiner. They used strategies similar to those in the first interview, but now they could more easily perform the required computations.

Sara is selling girl scout cookies. She has sold 4 boxes and has made $12.00. How much do you think each box of cookies is worth? Could you show how you think of this on this piece of paper?

Sara has met with some of her friends from the girl scouts and they are telling each other how many boxes of cookies they have sold and how much money they have made. To organize the information, they have built the following table. Could you help them complete it?

Boxes of Cookies	Price
1	
2	$6.00
3	$9.00
4	$12.00

7	$21.00
8	
9	$27.00
10	$30.00

N	

How much would one have to pay for 100 boxes? What do you think the n in this table means? How would you find the price if the number of boxes is n?

How do you get from one number to the other in the x column? How do you get from one number to the next in the y column? How do you get from the numbers in the x column to the numbers in the y column?

Figure 1: A table involving a 1 to 3 ratio.

How much are eight boxes? In the first interview 14 children (50%) found the price of eight boxes. Thirteen of these children produced 24 by adding three to 21 (the price for seven boxes), and one subtracted three from 27 (the price for nine boxes). In the second interview 22 children (78%) found the correct answer. They all computed 24 by adding three to the price of seven boxes. One child also stated that he could multiply three by eight to get to 24 and another mentioned that the price table was like the "times three" multiplication table.

How do the numbers change? Questions on this topic were not asked in the first interview. In the second interview, of the ten children who were asked to explain how the numbers changed down the number of boxes column, eight said that they increased by 1. For the changes in the price column, of the 11 children who were given the question, 10 stated that the numbers changed "by 3's" and one said that they changed by "adding 3's." For the relationship between the numbers across rows, only two children answered correctly. One child stated that for four boxes one would multiply 4 times 3, and the other explained that you count by 3's and "you stop when you get up to the number [of boxes] you have to get up to".

N and the cost of N boxes. The meaning of N and the cost of N boxes were dealt with in the second interview only. Eight out of 16 children answered that N could be "any number", a convention previously adopted in the classroom. Other children either did not answer or answered with a word with n as the initial consonant (e.g. nickel, none, nine). One child after discussing with the examiner how many times the number denoting boxes would go into the price of the boxes, stated that N boxes of cookies would cost 3 times N. Six children extended the table and stated that N was 11 and that the corresponding price was $33.

Problem 2: On Apples in a 2 to 3 Ratio

The second problem was more challenging, involving a 2 to 3 ratio:

Mary and Paulette are picking apples. For every 2 apples that Mary picks, Paulette picks 3 apples. (That is, every time Mary picks 2 apples, Paulette picks 3 apples.) How could you show this on this piece of paper?

Could you build a table like that in problem 1 to show how many apples Mary and Paulette pick?

Mary	Paulette

Figure 2: A table involving a 2 to 3 ratio.

In the first interview only six of the 25 children who were given this second problem completed the table. After writing 2 and 3 in the first row and 4 and 6 for the second, five children wrote the values for column 1 moving downwards by increments of two; they then worked on the values for column 2 by increments of three. Only one child worked row by row, concomitantly incrementing the values in columns 1 and 2. In the second interview, five months later, 16 children completed the table. Six children worked row-by-row and 10 children also worked with increments of two and three, but first completed column 1 and then proceeded to complete column 2.

What Next?

Given second graders' tendency to focus on the isolated columns' numerical patterns to complete function tables, our next question was: How could we, through instructional activities in third grade, help them to focus on functional relationships?

In a previous classroom intervention study (Schliemann, Carraher, and Brizuela, 2001) we found that third graders could correctly fill in function tables but, like the second graders we interviewed in Study 1, they did so with a minimal of thought about the invariant relationship between the values in the two columns. In that classroom intervention we made several changes in the structure of the table and the goals of the activities to discourage students from working on isolated columns. We found that, use of algebraic representations and a "Guess My Rule" game, with the number pairs presented in random order instead of following an increasing order, constitute an exciting context for children to break away from the isolated column strategies.

Use of algebraic notation helped them move from computations to generalizations about how two sets of values were interrelated.

In the study we report next, we explored how the children who participated in the interview study described above (Study 1) deal with linear functions as, in third grade, they were introduced to graphs of functions. The intervention, consisting of 16 ninety-minute classes throughout the school year, included work with function tables, algebraic notation, and graphs.

Study 2: Creating Contexts for The Graphical Representation of Functions

Graphs constitute symbolic conventional systems for representing changes in quantities and events. The conventions adopted in a graphical representation, although arbitrary, obey a coherent set of rules so that the spatial relationships in a graph are logically coherent with the relationships between the quantities or events it describes. In the case of function graphs a single line can represent the infinite number of possible number pairs that satisfy a certain function. The graph of a function visually captures the essence of a functional relationship. It also allows one to visually compare multiple functions.

In our longitudinal study on algebraic arithmetic we decided to focus upon graphs, instead of function tables or pairs of numbers. This was a bold decision given the difficulties many adolescents experience with function graphs. However, Piaget's extensive work on the child's conception of space (Piaget & Inhelder, 1948/1956) and geometry (Piaget, Inhelder, & Szeminska, 1948/1960) shows that the understanding of vertical and horizontal dimensions as a coordinate system can be achieved as the result of a long-term developmental process by age 9 or 10. From these results we could assume that most of our third graders would have some understanding about the coordination of vertical and horizontal lines in space that we could build upon. But we also needed to create the experiences that would explicitly relate spatial coordinates to numerical values, thus providing the initial ground for their understanding of graphs. We will next describe how we did this as we worked with three third-grade classrooms.

<u>A human coordinate grid.</u> During the first term in third grade we explored problems and representations of additive functions (Carraher, Brizuela, and Earnest, 2001). In the second term we focused on multiplicative functions and graphs. We first took the children to the gym and had them construct two parallel number lines, with points from zero to 12 about 1.5 feet apart. One line represented Karen's money and the other represented Franklin's money. Two children were asked to represent, by taking positions along the points in the lines, the relationship "Karen has twice as many dollars as Franklin". Thus, the child on Karen's line would always stand on a number that was twice the value of the position occupied by the other child on Franklin's line. Later, the two lines were positioned perpendicularly, meeting at the origin, thus constituting the x and y-axis of a graph space. Pairs of children took turns to represent the relationship between Karen's and Franklin's money on the perpendicular lines. After discussions about how the number taken by a child was related to the number for the other child, in terms of doubles and halves, we led the remaining children in the class to go, one by one, to the positions where imaginary lines drawn from the coordinate positions would meet. The children not yet called to assume positions on the graph space sat on the bleachers, watching and participating in the discussion.

A few children were at first uncertain about where they should stand and needed help from the instructor or from the other children. Others initially walked from the number on x-axis to the corresponding number on the y-axis. As the others reacted, they would correct their position, going to the imaginary intersection points. The positions on the graph were named (0,0), (1,2), (2,4), (3,6), etc. and each child taking a position on the grid received a piece of paper with his/her corresponding ordered pair written on it. A string was given to the child standing at the origin and extended across the space so that a straight line was "drawn" along the positions occupied by the children representing the other ordered pairs. The process was repeated with the relationship "three times as many".

Written representations. The following week we brought to the classroom a written representation of the graphing space in the gym. We displayed the statement "Karen has twice as many dollars as Franklin" and asked the children to draw a table of possible values of Karen's and Franklin's money and to find the places in the graph corresponding to each possibility. In the discussion that followed the instructor and the children referred to the positions specific children took in the larger scale graph. We then worked with the statement: "Ann has three times as many dollars as Franklin". We were surprised by how easily most children in the three classrooms could find the intersection points, and could say what the ordered pair at each point represented. Drawing the function line, however, was still an issue that not all of them seemed to have grasped. Many children randomly connected the points as if they were trying to draw a picture.

Candy bars and chairs. In the third week we asked the children to think about the intersection points in the graph space as places for tables in the gym. Each table would have a certain number of chairs where children could sit. On each table there would be a certain number of candy bars to be equally shared by those sitting at the table. We began by asking the children to locate in the graph the points corresponding to a table with 4 chairs and 4 candy bars and one with 6 chairs and 6 candy bars. They could easily do so and, after some discussion agreed that you would get the same amount of candy bar in either table. As Albert expressed it: "It wouldn't matter because if he [the child] went to [the table with 6 candy bars and] 6 people, he would, you would get one candy bar". The instructor asked whether there would be a better table, where one would get more candy than in the first two and Jessie chose, locating it in the graph, the table with 5 chairs and 6 candy bars. Erica stated: "I'd rather go to 7 to 1 table" and correctly located the coordinate (1,7) for the table with one chair and 7 candy bars. When the instructor asked them to show the worst table, the table to "send someone who misbehaved", Eric chose the point corresponding to the table with 7 chairs and 1 candy bar. Paul proposed that the candy bar on the table with 7 chairs could be a Hershey candy bar that comes with squares, easy to break apart. He drew a candy bar with 2 lines by 7 columns of squares, explaining that each person would get 2 out of the 14 small pieces he drew. Other children accepted his solution, implicitly considering 2 out of 14 as equivalent to 1 out of 7.

Throughout the activity the instructor led the children to represent the ordered pairs as improper fractions. The children's previous work in the gym was referred to in the discussion, giving meaning to the points in the graph and to the written representation of ratios. References to their everyday experiences with sharing helped the children but also led to choices that were not consonant to the mathematical model under discussion. For instance, as the class compared the points representing tables on the graph space, Paul considered that the 4 to 4 table was a better choice than the 1 to 1 table because "Someone may not want their candy bar" and Eric proposed that in the table with 2 chairs and 4 candy bars not everyone will get 2 candy bars because one person could get one and the other could get three candy bars. In these occasions, the instructor appealed to the idea of fairness in sharing the candy bars so that each child at a given table would have the same amount as the others. This constraint was accepted and used by the children as they solved the problem in Figure 3.

Table A has 3 candy bars and 2 chairs.
Table B has 6 candy bars and 4 chairs.
Table C has 5 candy bars and 2 chairs.

1. Show how you would divide the candy bars equally among those at each table.
2. Which table would you sit at? Why? Convince us that you chose the best table.
Are any tables the same?

Figure 3: Finding and comparing ratios

The children worked in groups and the instructor and other members of the research team helped them and asked them to explain what they were doing. After answering the questions, the children located the points corresponding to each table on a graph where the x-axis denoted the number of chairs or people sitting at each table and the y-axis referred to the number of candy bars on the tables. They used interesting strategies to distribute the candy bars, most of them concluding that those sitting at tables A and B would each get one and a half bars and those at table C were better off since they would get two and a half bars. Most of the children located the points corresponding to each table on the graph. Some of them correctly plotted the lines corresponding to different ratios and explained why they did so. Jennifer found the answers to the questions using the diagrams in Figure 4.

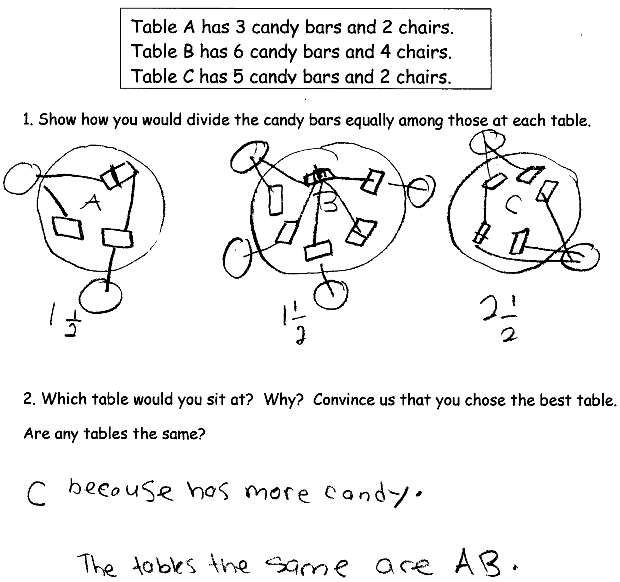

Figure 4: Jennifer’s work to determine the best table.

She then located the tables on the grid, at first mistakenly placing table A at (1,3). She then correctly placed tables A, B, and C, and an extra table D at (1,5) (see Figure 5).

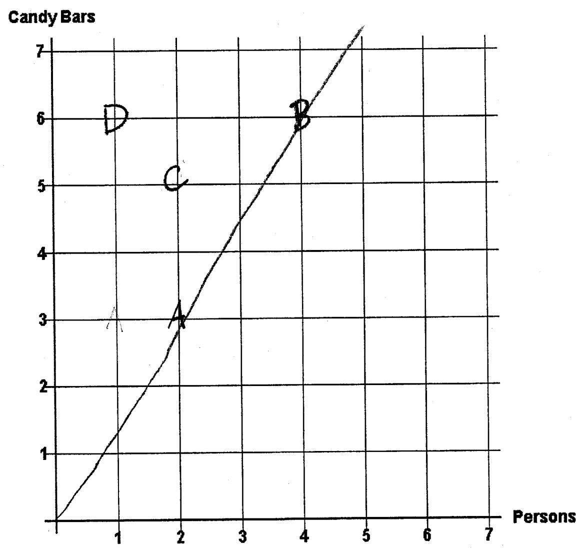

Figure 5: Jennifer’s graph. A, B, and C refer to the tables in the problem.

Before she drew the line linking tables A and B and the origin, Darrell asked her:

Darrell: What is the difference between tables A and B?
Jennifer: Nothing.
Darrell: Nothing? Are they the same thing?
Jennifer: Yeah.
Darrell: Oh. So do they fall in the same points in the graph, since they're the same thing?
Jennifer: No.
Darrell: How come?
Jennifer: Because, they're different but they're the same.
Darrell: Well, is there anyway that we can show on the graph, that they're the same? Even though they're on different points?
Jennifer: [Looks at the graph for a few seconds.]
Darrell: Let's think about it this way. Do you remember we were talking about, if there's one per-, the one, if there's one person and one candy bar, it's right here, there's two people and two candy bars, it's right here, three and three, and four and four, and five and five. And then we connected them?
Jennifer (nodding): Oh! They go on the same line.
Darrell: Why do you think they go in the same line?
Jennifer: Because they're the same.
Darrell: So, can you show me where that line would be?
Jennifer: [Draws the line starting at the origin, moving through points A and B.]
Darrell: Can I ask you a question? How come you didn't draw it like this- go up to A and then over to C, then over to D and then over to B?
Jennifer: Because they're not the same.
Darrell: They're not the same. Ah. So, which points will fall on this line?
Jennifer: The points that are the same.
Darrell: That are the same? And what does that mean, that they're the same?
Jennifer: Like, er-
Darrell: How much will each person get?
Jennifer: One and a half.

In this class children built upon their previous understandings of space and the specific experience with the large-scale graph in the gym. They also drew upon understandings about fairness, about how we share amounts, how the nature of the amounts to be shared determines how much you can do. They also benefited from the way classroom discussion developed as a natural conversation where different approaches were proposed and considered by the instructor and by the other children. In this sense, the discussion was much closer to an everyday problem-solving situation than to a traditional mathematics classroom focused on the transmission and application of rules.

Throughout this process the tension between everyday understandings and mathematics structures was constantly present, sometimes helping, sometimes hindering the analysis of pure mathematical relations. As the instructor explicitly acknowledged this tension the students were able to focus upon the mathematical relations, hopefully being introduced to some of the crucial issues involved in the interplay between mathematical models and everyday contexts. As the children became familiar with the conventions of graphs, they focused on the equality of ratios and on how the straight lines on a graph represent equal ratios.

Discussion

Even though second and third grade students would not easily focus on the functional relationships underlying data tables, they can learn significant things in classroom discussions and slowly work functional notation into their arsenal of representational tools. Our previous study on functional notation in the classroom (Carraher, Schliemann, & Brizuela, 2000; Schliemann, Carraher, & Brizuela, 2000) showed how algebraic notations used in the context of a Guess-My-Rule game led children to consider the functional relationship linking pairs of numbers. Our present study on the graphical representation of ratios shows that third graders can deal with the representation of points in a graph and that they can even understand how straight lines in a graph represent the same ratio.

In both classes children had access to the new representations we introduced, grounding them in previous experiences and ideas that gave meaning to them (Carraher & Schliemann, in press). But the most interesting feature of the studies relates to how algebraic and graphical notation constituted tools that helped them move from computations on isolated variables to generalizations about how two sets of values are interrelated.

We are still working to reach most of the children in the class and to expand their understanding and representation of mathematical properties in the domain of the multiplicative structures. The results up to now are rather encouraging and show, first, that most children in the elementary school grades can learn much more about mathematics that it is usually assumed and demanded from them. Second, it shows that the traditional sequence arithmetic first, algebra later can and should be replaced by a curriculum where the learning of arithmetic is part of activities involving algebraic concepts and representations from the start.

References

Carraher, D.W., Brizuela, B.M., & Earnest, D. (2001). The Reification of Additive Differences in Early Algebra: Vive La Différence! Paper presented at the Research Presession of the 79th Annual Meeting of the NCTM. Orlando, FL, April 2-4.

Carraher, D.W., Schliemann, A.D., & Brizuela, B.M. (2000). Early algebra, early arithmetic: Treating operations as functions. Plenary address. XXII PME-NA Meeting, Tucson, AZ, October 2000 [available in CD and at www.earlyalgebra.terc.edu].

Carraher, D.W. & Schliemann, A.D. (2001, in press). Is everyday mathematics truly relevant to mathematics education? In J. Moshkovich & M. Brenner (Eds.) Everyday Mathematics. *Monographs of the Journal for Research in Mathematics Education.*

Kaput, J. & West, M. (1994). Missing value proportional reasoning problems: factors affecting informal reasoning patterns. In G. Harel and J. Confrey (Eds.) *Multiplicative Reasoning in the Learning of Mathematics*. Albany, NY, State University of New York, pp. 237-287.

Nunes, T., Schliemann, A.D. & Carraher, D.W. (1993). *Mathematics in the Streets and in Schools.* Cambridge, U.K: Cambridge University Press.

Piaget, J. Grize, J.-B., Szeminska, A. & Bang, V. (1968/1977). *Epistemology and Psychology of Functions.* Dordrecht-Holland/Boston-USA, D. Reidel Publishing Company.

Piaget, J. & Inhelder, B. (1948/1956). The Child's Conception of Space. Atlantic Highlands, NJ, Humanities Press.

Piaget, J., Inhelder, B., & Szeminska, A. (1948/1960). The Child's Conception of Geometry. London, Routledge and Kegan Paul.

Schliemann, A.D. & Carraher, D.W. (1992). Proportional reasoning in and out of school. In P. Light & G. Butterworth (Eds.) *Context and Cognition*. Hemel Hempstead, Harvester-Wheatsheaf, pp. 47-73.

Schliemann, A.D., Carraher, D.W. & Brizuela, B.M. (2001). When tables become function tables. *Proceedings of the XXV Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands.

Vergnaud, G. (1983). Multiplicative Structures. In R. Lesh e M. Landau (Eds.), *Acquisition of Mathematics: Concepts and Process.* New York: Academic Press.