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Chapter 8

Is Everyday Mathematics Truly Relevant to Mathematics Education?

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Abstract

Early research in everyday mathematics lent support to diverse and often contradictory interpretations of the roles of schools in mathematics education. As research has progressed, we have begun to get a clearer view of the scope and possible contributions of learning out of school to learning in school. In order to appreciate this view it is necessary to carefully scrutinize concepts of real (as in "real life"), utility (or usefulness), context, as well as the distinction between concrete and abstract. These concepts are crucial for determining the relevance of everyday mathematics to mathematics education; yet each concept is deeply problematic. The tension between knowledge and experience acquired in and out of school is not a topic of mathematics. But it deserves to be a fundamental topic in mathematics education.

Introduction

Over the years we have witnessed a wide range of reactions to everyday mathematics research. We have seen teachers awed by street vendors solving problems. On the other hand, we have seen mathematics educators question whether serious mathematics was being learned out of school and whether semi-skilled trades encourage the development of advanced mathematical concepts. For some the results suggest that mathematics does not require formal instruction; for others the results merely reaffirm the importance of schooling; for still others, the results call for educational reform focusing on how people think, communicate, and learn out of school.

When people interpret the same data in such diverse ways, are they simply projecting their prior views onto inkblots of data? When they agree, are they simply celebrating common beliefs? Can we make headway in understanding the relevance of everyday mathematics for education? Or will our conclusions ultimately reflect our pre-existing educational agenda, our socio-political persuasions, and our views of learning?

We share with the other contributors to this monograph a number of premises about the nature of mathematical learning. We view mathematics, for example, as both a cultural and personal enterprise. It is cultural because it draws upon traditions, symbol systems, ideas, and techniques that evolved over the course of centuries, having originated in human activities such as surveying, astronomy, building, commerce, and navigation and eventually becoming a partially autonomous field of endeavor with its own subject matter, purposes, tools, and concerns. It is personal insofar as it demands from learners constructive processes and creative rediscovery even when they are apparently engaged in mere assimilation of facts and conventions.

We also share a number of ideas and beliefs about the purposes and spirit of mathematics education. We tend to distrust rote, unquestioning learning. We favor situations in which students themselves must decide how to frame and represent problems, but we realize that, left to their own devices and inventiveness, students are not going to discover many important concepts of elementary mathematics. We value problems that promote engaging discussions among students. We recommend that students and teachers pursue multiple paths of reasoning when solving problems. We encourage educators to make frequent comparisons between everyday language and symbolism and the formal language and symbolism of mathematics. These value premises are rooted in our personal backgrounds, in the economic and political Zeitgeist, and in our theoretical lenses as well as in the legacies of Plato's Meno, Rousseau's Émile and Piaget's constructivism (even though they may at times push and pull in somewhat opposed directions). Some hallmarks of these influences include the belief that mathematical concepts are "out there" to be discovered but require the careful questioning and guidance of the more learned (Plato), faith in the capabilities of children to learn provided we are sensitive to their views and motivations (Rousseau), and the importance of painstakingly documenting children's reasoning and handling of mathematical invariants as well as the need to place (mathematical) learning in a long-term developmental perspective (Piaget).

Although discussions about everyday mathematics take place in value-laden contexts, there is still plenty of room for scientific work. As the number of studies of everyday mathematics has grown, we now are beginning to understand the range of mathematical concepts likely to develop (or not) in out-of-school environments. Studies of classroom contexts are providing much needed data on how out-of-school knowledge plays a role in classroom problem-solving. This chapter represents an attempt to stand back from the data and try to make sense of what we have observed and learned. In doing so, we will try to make clear how we have had to adjust our beliefs about the relevance of everyday mathematics as we learned more from research. The present chapters in this monograph provide a rich set of empirical data and discussions that will help to refine and clarify educators' views and beliefs on the educational relevance of everyday mathematics.

What Does Research on Everyday Mathematics Suggest?

Our views of the relevance of everyday mathematics for mathematics education developed over the years as a result of our research on the characteristics of everyday mathematical understanding as well as students' understanding of school mathematics (see reviews by D. W. Carraher, 1991; Nunes, Schliemann, & Carraher, 1993; Schliemann, 1995; Schliemann, Carraher, & Ceci, 1997). Studies such as the ones in this monograph are crucial to the discussion and refinement of those views. What lessons do they offer us about how mathematics is learned or ought to be learned? What do they say about the relevance of informal mathematics to mathematics education? In view of the discussions and data provided in this chapter, we will try to reflect upon what we consider to be the main issues raised by them trying, at the same time, to enrich our own views concerning everyday mathematics and mathematics education.

Learning In and Out of School

Everyday mathematics research has repeatedly produced evidence that people learn mathematics outside of school settings. Specific cultural activities such as buying and selling promote the development of mathematical ideas that were previously thought to be only acquired through formal instruction. Our research shows that groups with restricted schooling master arithmetical operations, properties of integers and of the decimal system, and proportional relations (Nunes, Schliemann, & Carraher, 1993). Besides arithmetic (T. N. Carraher, Carraher, & Schliemann, 1982, 1985; Saxe, 1991; Lave, 1977, 1988), concepts and procedures related to measurement (T. N. Carraher, 1986; Gay & Cole, 1967; Saraswathi, 1988, 1989; Saxe & Moylan, 1982; Ueno & Saito, 1994), geometry (Abreu & Carraher, 1989; Acioly, 1993; Gerdes, 1986, 1988; Harris, 1987, 1988; Millroy, 1992; Schliemann, 1985; Zaslavsky, 1973), and probability (Schliemann & Acioly, 1989) are used in everyday settings by children or adults with little access to school instruction.

In retrospect, the observation that mathematical learning occurs out of school may seem obvious. Indeed one might wonder how anyone could have ever thought otherwise! After all, commerce and crafts requiring rudimentary measurement skills have often flourished in societies where schooling has been infrequent or even nonexistent. Furthermore, developmental psychological studies, particularly those of the Piagetian tradition, have long since documented that young children discover, for example, the commutative nature of addition before entering school.

Some remarks can help clarify how out-of-school mathematical learning could evoke surprise from researchers. Firstly, and this is particularly true of our own research on mathematics of street vendors in Brazil, we were surprised by the fact that the very same people presumed to be unskilled in mathematics could solve problems in out-of-school settings. When we presented the same street vendors with (what we thought were) the same problems in a school-like setting, they tended to find them confusing and gave answers suggesting they were having trouble with the basic sense of the problems. For example, they would sometimes claim that the amount of change to be returned to a customer after a purchase would be greater than the amount of money originally handed to the seller!

Looking back, it seems that we were not asking street vendors to solve “the same” problems at all. By indirectly suggesting in the school-like setting that they should write out their answers, we induced them to use poorly understood computation routines that involved features, such as borrowing from neighboring columns, that they did not employ in their work as vendors.

But how were they solving problems in their own ways? This brings up a second source of our surprise. If they were not following school-prescribed routines, but nonetheless producing correct answers, they must have alternative ways of representing and systematically solving problems. Much of our work in everyday mathematics pursued this question. It now seems fairly clear that many of the street vendors did not use a place value notational system when mentally solving problems. Furthermore, they seemed to operate on measured quantities (such as 3 coconuts, 35 Brazilian <u>cruzeiros</u>) as opposed to pure numbers (3, 35). In this manner they did not have to perform calculations on numbers and introduce the result of the computation back into the meaningful problem context. Rather, they would always be working directly with countable quantities.

As we came to document more and more instances of everyday mathematics—among carpenters, farmers, lottery bookies, and construction foremen—the more we realized, spurred by suggestions from Resnick's (1986) work, that alternative mathematics to be useful at all would have to pay heed to some basic properties of arithmetic as additive composition and the commutative and associative laws of addition. Once placed in this framework, we began to see informal mathematics and the mathematics of the school as more closely related than we had originally thought. They both had to respect many of same basic properties of arithmetic, such as the associative law, but they often did so through distinct routes. For example, when subtracting 135 from 200, a street vendor might take away 100, then 30, then 5. This strategy relies on the decomposition of 200 into 100 + (70 + 20 + 10); likewise, 135 is implicitly treated as the sum, 100 + 30 + 5. Although the street vendors did not know how to explicitly express the associativity of addition they revealed their implicit use of the property through the transformations they made on the values given. The standard school algorithm for column subtraction invokes the same general property but decomposes the givens in a somewhat different manner. This quick overview does not do full justice to the subtle differences between arithmetical algorithms in and out of school. We should point out, however, that many street vendors appeared not to represent values through a place value system as our reconstruction of their thinking may mistakenly suggest.

The dramatic contrasts we encountered among Brazilian street vendors predisposed us to view informal mathematics as inherently more natural and meaningful than school mathematics. We showed that people attempt what appear to be nonsensical solutions in a school-context while they search for meaningful solutions when the problems are part of their work in everyday contexts (see T. N. Carraher, Carraher, & Schliemann, 1985, 1987; Grando, 1988; Lave, 1977; Reed & Lave, 1979; Schliemann, 1985; and Schliemann & Nunes, 1990). The analysis of problem-solving solutions in and out of schools suggests that students commonly learn algorithms for manipulating numerical values without reference to physical quantities, only reestablishing clear links to the problem context in the end when the units of measure are finally attached to the numbers. By contrast, individuals solve problems in the workplace using mathematics as a tool to achieve goals that are kept present throughout the solution processes with continuous reference to the situation and the physical quantities involved. As such, the problem solvers in the workplace are normally aware of how the quantities generated in the course of the computations are related to the problem at hand.

We also stressed that schools encourage memorization and repetitive practice, whereas at work street sellers solve problems through mental computation, using flexible strategies they develop and efficiently apply to achieve their selling goals. These mental computation strategies are based on an understanding of basic logico-mathematical relations such as the properties of the decimal system and of linear functions.

Is it possible that we overstated our case? Perhaps there is a tradeoff between meaning and power. Perhaps to get things done efficiently we must sometimes forge ahead using specially designed tools, the inner workings of which are not fully understood. Long division is a case in point.

Schools move relentlessly away from the here and now, pushing students to adopt techniques and consider properties by no means directly suggested by the original problem contexts. This is likely to lead to "teething" problems from time to time; namely, some discomfort with the newly introduced objects. Unfortunately, many students never get over this state of discomfort, because educators frequently forge ahead when students correctly master the techniques.

We wonder whether all mathematical knowledge should be grounded in deep understanding, by an ability to systematically relate the matters at hand to a broad range of concepts and instantiation in diverse contexts. Should students only move on to new topics when they have thoroughly understood what comes before? Or should the teacher immerse them in complex situations and allow them to struggle for meaning?

Observations from everyday mathematics do not provide straightforward answers. Consider the following description by Guberman (1999) of an 18-month-old girl purchasing candy in a Brazilian shantytown:

(She) approached the stand holding a Cr$100 [one hundred cruzeiro] note in her hand. Another customer at the stand held her up to the counter so she could see the merchandise, and she pointed to the type of candy she wanted. The owner of the stand gave her the candy and took her Cr$100 note, and she headed back home having completed a fairly complex commercial transaction without uttering a word and without engaging in any mathematical calculation. The toddler was able to participate in this exchange--a transaction beyond her independent ability--because others had structured the task for her; in this case, her mother knew what candy the toddler would select and had given her the exact amount of money she needed. (Guberman, 1999, p. 213).

This lovely example shows a child becoming socialized in the logic and meaning of commerce. She probably has only a vague idea of what money is. She has little if any idea of the precise cost of the candy she wants, much less of its cost relative to other goods at the outdoor stand. Yet she is learning several things as she makes a purchase of candy "on her own." She understands that the vendor has in his legitimate possession an item that she would like to have, that if she gives him some special paper called money and points to the desired item, he will give it to her. In subsequent interactions she will no doubt learn that the man can refuse to give her what she wants if he believes she is not giving him enough money in exchange. She will also learn that some pieces of money are worth more than others, that occasionally some money will be returned to her, possibly not from the money she handed over.

We could argue that the everyday example demonstrates learning in a meaningful situation. But we should also recognize that the child is being immersed in what is initially a partially understood social transaction. There may be things she is fairly aware of, for example, that the man will not harm her in her mother's presence.

But there are other things she will no doubt find mysterious; for example, why the man could be willing to trade some delicious candy for a piece of paper. Only later may she realize that he too can obtain things he desires from the piece of paper she gave him.

So informal learning does not distinguish itself from school learning by virtue of participants being swaddled in familiar situations. The unfamiliar lurks both in and out of school. And in neither setting are participants given full explanations before becoming immersed in the problems. Just as we talk to infants before they know what the words mean, we engage children in learning situations, both in and out of school, before they fully understand what they are partaking in. This is one of learning's many peculiarities.[1]

Realism and Utility

In common parlance, problems are realistic to the extent that they typify those encountered in mundane life situations. On this basis Thorndike (1922) criticizes textbook problems requiring students to solve problems with Roman numerals or with numerical values unlikely to appear in accounting situations. Realism is thus closely allied with the value of utility (what do they need it for?). Most people are staunch defenders of realism and utility in mathematics education. But are realistic situations the best path to solid and flexible mathematical learning?

The studies in this monograph show that when everyday tasks are brought into the classroom they are not approached in the same way as they would be if children were outside of schools. Brenner and Moschkovich's chapters emphasize that the way teachers present their assignments in the classroom, more than the realistic aspects of the task, plays an important role in children's problem-solving approaches. Civil's data also raise questions concerning each child's level of engagement and understanding while participating in realistic tasks. And in Moschkovich's study we see how workplace practices may not always coincide with academic mathematical activities. In the classroom one solves and discusses problems demonstrating knowledge to the teacher and other students. Success or failure in solving a problem brings consequences, but these consequences are different from those incurred out of school.

Civil's observations of children discussing the tiling of geometric figures provide an interesting example of children heavily engaged in what some might consider an unrealistic task. Should such discussions be discouraged in the classroom because children are unlikely to ever be faced with a tiling problem later on in life? Or is it possible that some other purposes are being met by such activities? Should all learning situations pass the test of realism? If so, how do we define realistic problems for different possible occupations? What is a realistic problem for a future pure mathematician? Or meteorologist? Or graphics designer? Or marriage counselor?

What makes everyday mathematics powerful is not the concreteness of the objects or the realism of the situations dealt with in everyday life, but the meaning attached to the problems under consideration (Schliemann, 1995). And meaningfulness must be distinguished from realism (D. W. Carraher & Schliemann, 1991). It is true that engaging in everyday activities such as buying and selling, sharing, or betting may help students establish links between their experience and intuitions already acquired and topics to be learned in school. However, we believe it would be a fundamental mistake to suggest that schools attempt to emulate out-of-school institutions. After all, the goals and purposes of schools are not the same as those of other institutions.

Utility (Bentham, 1962/1789) is no less slippery a concept than realism. Nothing appears more self-evident and immune to criticism than the value of utility. To demand that schools teach what students will later use in life seems to be the very least we can hope from education. Or, as students themselves sometimes express it: "If we're not going to use it, why do we have to learn it?"

Consider the case of an eighth-grade peasant child from a third-world country. The child's parents and teacher may seriously wonder why she is required to study algebra. After all, she doesn't need to use algebra to raise a few farm animals and plant corn and cassava root. The last thing they want to do is send her to school, year after year, to fill her head with knowledge she will never use. Their concern is predicated on the beliefs that (1) the girl will continue to exercise the occupation of her parents, (2) algebra is irrelevant to running a small farm, and (3) that only knowledge that can be immediately and directly applied is worthwhile. Each of these beliefs merits consideration.

What if the child should choose to become an engineer or hospital laboratory technician? Could she really hope to meet the requirements of these careers several years later if she does not begin to establish a strong mathematical background in algebra now? Or should others decide that this option is not a realistic or desirable one for her?

Regarding the third point, some would argue that a broad, diversified educational background is the most useful sort of preparation schools can offer. Not because one learns directly applicable facts and techniques, but rather because one learns how to make sense of ever changing situations, to establish priorities, to place problems in broader contexts, and to generate wide ranges of options when faced with highly constraining conditions. Perhaps the most useful of tools are not prêt a porter. This claim accepts the utilitarian view that knowledge should be useful; it reminds us, however, that not all knowledge, and perhaps not the most important knowledge, is immediately and directly useful.

Contexts Given and Taken

The ease with which one employs the term, "context," can be deceptive. Contexts appear to be given, as in the expression "the historical context of World War I" or "the context of the Brazilian economics in the mid-60s." Likewise, in mathematics education we often speak of problem contexts as being given by, contained in, or implied by word problems. A word problem, such as “Martin had 20 baseball cards; he sold 17; how many does he have now?” for instance, would refer explicitly to a familiar commercial context. The numbers are associated with physical entities, baseball cards. These are familiar to most (American) students who know why they are collected, what information they contain, and how they relate to the sport of baseball. They also know a lot about the nature of buying and selling. The word problem could presumably be contrasted with a “context-free” problem such as 20-17 = _. Proponents of "contextualized mathematics" commonly claim that the former sort of problem is inherently richer than the latter. Further, they might argue that if a teacher were to stage a commercial transaction in the classroom, with some students acting as sellers and others as buyers, the students could immerse themselves even more deeply in the problem. This highly contextualized problem would then provide students with a richer, lifelike set of experiences. For these and other reasons, contextualized mathematics proponents would tend to favor the school-store approach over the context-free, artificial problem denuded of a meaningful context. We ourselves embraced this sort of reasoning in our research with Brazilian street vendors (T. N. Carraher, Carraher, & Schliemann, 1985) and in an experimental study of school children (Carraher, Carraher, & Schliemann, 1985, 1987) who answered "contextualized" and "decontextualized" word problems. But we have come to realize that there are serious shortcomings to this view.

One of the corollaries of a “contexts as given” approach is the belief that abstract symbols are less rich and meaningful than pictures and specific concrete instances of real-world objects and events. Thus, written symbols such as =, +, π, ∑, a, 1, 2, and their concatenation in expressions such as $y = 2x + b$ are presumably more abstract, and therefore less meaningful, than everyday objects and statements such as “if you pay twice as much, you get twice as many tomatoes.” No one would dispute this for the case of mathematically illiterate people for whom mathematical notation may appear to be nothing more than scribbling. But what about for mathematicians and for students?

In recent classroom research with third-grade students in greater Boston (D. W. Carraher, Schliemann, & Brizuela, 1999), we began to explore diverse manners of representing the addition and subtraction of quantities. After the first class, we asked the students to solve the following problem as homework and to draw a diagram showing that their solution was correct.

Marcus had 17 goldfish, but 6 of them died. How many goldfish does he have now?

Two general sorts of solutions emerged. Some children wrote the expression, "17 – 6 = 11." Others drew iconic or semi-iconic representations of the fish. Such drawings preserved the fish-character of the quantities. Sometimes they represented the initial quantity and the quantity to be subtracted as in Figure 8.1. In other cases, they drew the initial quantity (17 fish) and then marked the 6 dead fish in some way (e.g.,used x's to cross out the dead fish) to distinguish them from the live fish (see Figure 8.2).

Figure 8.1 here

Figure 8.2 here

Although these representations showed that the children could make meaningful representations of the relations among the quantities, we were committed to helping them to go beyond this. We wanted them to move towards thinking of numbers and quantities as continuous, uncountable entities, and we knew that the number line would eventually become an important representational tool for them. For these and other reasons we began to introduce representations of quantities as (from our point of view) directed line segments. Line segments were generally consistent with number-line representations of magnitudes insofar as both expressed order through length.

After reviewing the students' "spontaneous" drawings of the fish problem, we showed them the following diagram and asked them whether it could apply to the case at hand.

Figure 8.3 here

The students quickly related the fish problem to the vector diagram in apparently meaningful ways. They interpreted the uppermost vector as the original quantity (17 fish), the middle vector as the quantity lost (the fish that died), and the lowest vector as the result (11 fish).

We then asked them if someone else looking at the vector diagrams would know that it concerned the problem of fish. They immediately and enthusiastically supplied alternative problem contexts for the same diagram. One girl suggested that the arrows could stand for gum and that the middle arrow represented gum that was chewed. A boy submitted that the backwards arrow could be like the action of taking away blocks. Another girl said that the arrows could represent stars in the sky that appear and disappear. A boy remarked that the diagram could represent 17 monsters of which 6 ran away.

This simple example illustrates that children can appropriately bring contexts to bear on situations even when those contexts are not directly given by or contained in the problem. Furthermore, since the interpreters can provide the contexts, symbols can be meaningful and useful without being explicitly bound to specific, concrete contexts. Indeed, one of the advantages of symbols such as arrows derives from their very lack of reference to specific quantities; iconic drawings may not be nearly as useful in this regard.

Providing contexts and entertaining new contexts introduced by others is an important part of mathematical reasoning. Arcavi (in this volume) makes a similar point and we agree with the thrust of his analysis regarding contextualization. But the relationship between contextualization and mathematization deserves closer attention.

Symbols and representational systems are abstract not because they are removed from contexts, but rather because they can be employed in a very wide range of contexts. That is to say, abstraction and concreteness are not mutual alternatives but intertwining concepts. One way a person can show that she understands an abstract relation is by exemplifying it through multiple examples in diverse contexts.

This point relates directly to Cassirer's (1923/1953) criticism of the Aristotelian notion of universal (read "abstract") concepts. Cassirer makes the point that "what was beyond all doubt [in Aristotle's logic] …was just this: that the concept was to be conceived as a universal genus, as the common element in a series of similar or resembling particular things" (p. 9).

Cassirer realizes that the classic notion of abstraction presumes that more general classes and concepts (e.g., number) would be virtually devoid of content were we to insist upon the idea that they consist of the common elements of more particular instances (integer, imaginary number, rational number, transfinite number, algebraic number, etc.). Abstract concepts would be largely impoverished and devoid of meaning if they consist of the shared elements or properties of lower-level, more concrete concepts.

He proposed as an alternative to the "common or shared elements" approach the idea that more abstract concepts subsume, rather than exclude, the particular properties of subordinate concepts and categories. Throughout Substance and Function, he lays the groundwork for a refreshingly different view about the relationship between concrete and abstract:[2]

The genuine concept does not disregard the peculiarities and particularities which it holds under it, but seeks to show the necessity of the occurrence and connection of just these peculiarities. What it gives is a universal rule for the connection of the particulars themselves. Thus we can proceed from a general mathematical formula,—for example, from the formula of a curve of the second order,—to the special geometrical forms of the circle, the ellipse, etc., by considering a certain parameter which occurs in them and permitting it to vary through a continuous series of magnitudes. Here the more universal concept shows itself also the more rich in content; whoever has it can deduce from it all the mathematical relations which concern the special problems, while, on the other hand, he takes these problems not as isolated but as in continuous connection with each other, thus in their deeper systematic connections. The individual case is not excluded from consideration, but is fixed and retained as a perfectly determinate step in a general process of change. (pp. 19-20)

Abstract thinking is not antagonistic to the idea of reasoning in particular contexts. As Arcavi (in this volume) showed in his investigation of number palindromes, the bus context inspired him to think about palindromes as packed in a roll of bus tickets. This in turn favored his thinking about palindrome density in ways different from students who took the problem as a textbook example of digit permutations. Abstract thinking does not preclude thinking which is tied closely to particular, even concrete, contexts. It was precisely in this spirit that we proposed the expression "situated generalization" (D. W. Carraher, Nemirovsky, & Schliemann, 1995) several years ago.

But it is important not to think of contexts solely in terms of physical or social settings and their constraints. To do so would deny the existence of mathematical contexts. But what are mathematical contexts if they are not grounded in physical reality?

Recontextualizing Problems

Several years ago the first author became intrigued by the well-known Goldbach conjecture.[3] This conjecture can be simply stated:

Any even number greater than two is the sum of two primes.

Goldbach conjecture: classic formulation in natural language

Although it has never been disproved (no one has ever found an even number that cannot be written as the sum of two primes), no one has proved it. Nonetheless it has given rise to some important work in number theory. Our purpose in reviewing here one person's (the first author's) thinking about the problem is to explore the nature of mathematical contexts.

Thinking about the Goldback Conjecture

If *n* corresponds to any positive integer greater than 2, the conjecture is really a statement about every even integer (i.e., 2n) being composed of p plus q, where p and q are prime numbers (positive integers greater than 1 and divisible only by themselves and 1). That is,

For all *n*>2, there exist p, q such that 2n = p+q, where *n* is an integer, and p, q, are primes.

Goldbach conjecture: formulation in mathematical notation

The metaphor, which initially appealed to me, was that of a necklace of numbers. Each integer, *n*, can be thought of as a bead on that open necklace, the first bead representing 1, followed by a second bead representing 2, and so on. In the necklace model, the conjecture corresponded to the idea that

It is possible to find a point between two beads along any "even" necklace that will split it into two segments, each of which contains a prime number of beads.

Goldbach conjecture: Necklace formulation

From time to time over the course of several days, I tried to understand what it would mean to have a prime length of beads. Since I knew that primes were those numbers that remained after all multiples of other numbers were cast out, I spent some time working with "number necklaces," which I instantiated through grid paper in which each small square stood for a bead. An even-numbered sequence of squares could be taken to represent any open necklace with 2n beads. Starting from the left and crossing out every second bead, then every third bead, every fifth bead, and so on (multiples of 4 need not be stricken since they are included in multiples of 2), I was essentially applying a sieve of Eratosthenes to the squares-beads-numbers. This procedure leaves untouched (i.e., not crossed out) all possible prime lengths for the left segment. Using the same procedure from the right side, I was able to produce prime lengths for the right segment. This activity helped me recontextualize the Goldbach conjecture in (what was for me) a new way. I now understood the conjecture as stating that I would always find a point between two beads for which the bead just before the cut would have a prime ordinality (that is, would remain after the sieve of Eratosthenes was carried out when counting from the left), and the bead just after the cut would have a prime ordinality (from the right).

It occurred to me that a successful cut-point (there often was more than one place to divide a necklace into two prime lengths) would be a certain integer distance from the midpoint of the necklace. Sometimes a cut-point would lay exactly at the midpoint of the necklace, as for necklaces of length 14, 26, and 38 beads, each of which was double a prime. But I noticed that any successful displacement away from the midpoint (e.g., in the case of a 16-bead necklace, 8+8 is not a solution; however, 11+5 is a solution as is 13+3), the amount by which one of the primes *surpassed* the midpoint was equal to the amount by which the other prime *fell short* of the midpoint. For example, offsetting the cut 3 units from the midpoint of a 16-bead necklace yielded 11 beads in one segment and 5 in the other; offsetting the cut by 4 units would result in a [13, 3] partition. The increment of beads to one-half of the necklace was always the same as the decrement from the other half.

This insight inspired me to express the prime lengths, p and q, in a new way, namely as displacements from the midpoint. Now, since the midpoint of a necklace 2*n* units long split the necklace into two segments, each *n* units long, *p* and *q* could be rewritten as $n + d$ and $n – d$. So the conjecture could be restated as a statement about two sums:

For all integers greater than 2, there exists an integer *d* (for integer displacement, including zero) such that $(n+d) +(n –d) = 2n$, where p (∈ Primes) = $n+d$, and q (∈ Primes) = $n – d$

Goldbach conjecture formulated in terms of displacements from a midpoint

I knew from elementary algebra that (n+d) and (n-d) looked very much like the factors (x+c) and (x–c) of an equation of the form x^2-c^2 and decided to multiply them. In my notation the result of the multiplication was n^2-d^2. So this eventually led me to the equation,

$pq = n^2-d^2$

Goldbach conjecture formulated in terms of the product of two primes and differences between squares

What was this expression to mean in terms of the Goldbach conjecture? Well, the original statement about the sum of two primes was now transformed into one about the product of two primes! This can be expressed as “For all integers, *n*, greater than 2, there exist primes, p and q, and an integer, *d*, such that $n^2 = pq + d^2$." In other words, all squares (of integers greater than 2) can be expressed as the product of two primes plus a square (of an integer).

This formulation got me thinking about a new geometrical way to represent the conjecture. What would it mean to draw n^2 and d^2 as squares and pq as a rectangle? It would mean that

> For any square of integer length side we can always find another square of side *d* which, when removed from the first square, will produce a region equal in area to a rectangle with sides *p* and *q*, where *p* and *q* are primes.

Product of primes formulated in terms of figures in a plane.

This formulation begged to be given a geometrical interpretation. After sketching a series of solutions (workable squares and rectangles) onto grid paper, I decided to represent this most recent formulation in a coordinate space. Since only integer values of x, y mattered, this space turned out to be a unit lattice. A square of side *n* could be represented by the region defined by two corners, the origin (0, 0) and the point (*n*, *n*) lying on the line y = x. A solution rectangle of sides, p, q, would be defined by the origin and the point (p, q). But we already noted earlier that p and q were equidistant from *n*. In the coordinate system, this meant that, for instance, if p were 3 units greater than *n*, then q would have to be 3 units less than *n*; the increment for one was the decrement of the other. Now, as I started incrementing and decrementing from the point (*n*, *n*), it occurred to me that this was equivalent to taking *d* unit-steps down from the point (*n*, *n*) and *d* steps rightward from (*n*, *n*), leaving us at point (*n*+*d*, *n*-*d*). Alternatively, it could mean taking *d* unit steps up from (*n*, *n*) and *d* steps leftward from (*n*, *n*) to arrive at the point (*n*-*d*, *n*+*d*). When I thought of all of the unit displacements that could be made, it occurred to me that they would always lie on a diagonal line perpendicular to the line y = x, on which (*n*, *n*) resides. What should we call the new line that contained the results of the displacements? With some struggle I worked out that this line began at point (0, 2n) and continued downward with a slope of –1 to (2n, 0). This new line also contained the point (*n*, *n*) itself. Once these facts were known, I was able to reach the conclusion that the line must be y = 2n –x. This gives rise to a surprisingly new formulation of the conjecture:

> For all integers, *n* >2, there is at least one pair of values (x_1, y_1) satisfying the equation y = 2n –x, such that x_1, y_1 are primes.

Goldbach Conjecture as a solution to a first-order Diophantine equation

If we think of y = 2n –x as a family of diagonal lines in Quadrant 1, each of which has a slope of –1, then the conjecture states that each of these lines has at least one point (p,q) where the coordinates are primes.

This may seem foreign to the earlier formulations. However, it merely expresses the conjecture in different contexts. Certain relations can be noted across the representations. If we try to relate the Diophantine equation representation to the necklace model, we can understand that the necklace corresponds to any of these diagonal line segments, for example, y = 2(6) – x, which begins at (0, 12) and ends at

(12, 0). A solution for this case is (5, 7) and by symmetry (7, 5). This corresponds to a 12-bead necklace split into lengths of 5 and 7 beads. Although it might at first appear that each coordinate in this line corresponds to a bead, the intersections really correspond to cut-points, that is, points between two beads rather than beads; this occasionally leads to some confusion in moving from the necklace to the coordinate/unit-lattice model. Furthermore, the Diophantine formulation, $y = 2n - x$, is really just a restatement of the initial conjecture, $2n = p + q$. We have now returned to the point of departure of the problem.

These reformulations do not make the Goldbach conjecture look any more true or false than it did when we started. Yet I now can understand it better than before. I can place it in several mathematical contexts that I did not envision when I first looked at the problem. These contexts were certainly not imposed by the immediate physical constraints of the problem. Yet neither were they free-flowing inventions or Platonic reminiscences. The necklace metaphor allowed me to get an initial grip on the problem and explore how solutions manifest themselves within that metaphor. This ultimately highlighted the issue of length, and since the issue was now one of partitioning the length into two parts, it became natural to consider the midpoint of the necklace, for which the parts are equal, as a point of departure. The midpoint would always be the average of the solution-primes as well as half the length of the necklace. Focusing on the displacement between the midpoint and the solution lengths eventually led me to a notational formulation that conjured up long-ago studied information about the algebraic factoring. This in turn opened the door to thinking about products, one bona fide use of which lies in computing the areas of quadrilaterals.

Were these contexts real and meaningful? Yes, to me they were as real as deciding upon the best buy in a supermarket, even though it didn't particularly matter what setting I was in I was when I was trying to solve the problems. From time to time, how the problem was represented seemed to make a difference. For example, when I set the problem onto grid paper, it was a short step to placing the problem in the context of coordinate geometry. This in turn made it very natural to look for notation from linear algebra that would clarify the properties I was beginning to take cognizance of in this new representation.

Our View of Contexts

Research sorely needs to find theoretical room for context not reducible to physical settings or social structures to which the student is passively submitted. Contexts can be imagined, alluded to, insinuated, and explicitly created on the fly or carefully constructed over long periods of time by teachers and students. Much of the work in developing flexible mathematical knowledge depends on our ability to recontexualize problems, to see them from diverse and fresh points of view, to draw upon our former experience, including formal mathematical learning. Mathematization is not to be opposed to contextualization since it always involves thinking in contexts. Even the apparently context-free activity of applying syntax transformation rules to algebraic expressions can involve meaningful contexts, particularly for experienced mathematicians. (It is ironic that the mechanical following of algorithms characterizes the approaches of highly unsuccessful and highly successful mathematical thinkers.) We may not always recognize the contexts because we do not share the same knowledge and experience of others. And since contexts are not fully constituted by their physical properties, but always involve issues of meaning and interpretation, we cannot assume that children sharing the same physical settings as ourselves will be interpreting problems in the same contexts as we do. This is not to say that it is impossible to establish true communication between teachers and students. Rather, it means that one of the challenges in teaching mathematics is to help children recontextualize problems and issues.

Sometimes this will require drawing upon experiences and knowledge children have acquired in everyday settings. For example, one of the deeply rooted notions children appreciate from everyday life is that a total amount consists of the sum of its parts, none of which is greater than the whole. They also know that things contained cannot exceed the limits of a container. These ideas are closely related to Euclid's fifth common notion or axiom in Book I of the Elements, namely, that "The whole is greater than the part" (Heath, 1956, p. 155). Such a belief can be useful time and time again in working with numbers and quantities. However, there are times in one's mathematical education when it must be subjected to scrutiny and modified. For example a vector sum may consist of a number of components, one or more of which can be greater than the resultant vector, the whole.

Vectors confront children with perplexing results, such as the fact that adding one positive unit to a vector two negative units in length produces a result, a total that is shorter than the negative-two unit segment but nonetheless represents more than was earlier. We analyzed this very problem with several fifth grade students a few years ago (D. W. Carraher & Schliemann, 1998). In the following example a student eliminates the paradox—she and her colleague had been initially puzzled by the results--by making use of what she has learned about the number line (values to the left of zero are less than nothing) as well as from drawing upon their experience with money which involves both positive values (credits) and negative values (debits):

Talulah: Negative one U is more than negative two U's. Even though, when you look at it, negative two becomes [is] bigger. Two spaces is bigger [than one space] but it is closer to a positive number, and positive numbers are bigger than negative numbers.

When she carries out the addition twice, the resultant vector is zero-length, to which she replies:

Talulah: Yes, even if you can't see anything, you have more [than before] because …now you don't have any money but you are out of debt.

She recontextualizes the puzzle before her by reinterpreting it in light of what she has learned in school and out of school. The context of the problem has become redefined. What was originally about actions taking place on rectangles (or, perhaps, line segments) on a computer screen gets recast as an example of how a person slowly emerges from debt.

Can Everyday Mathematical Knowledge Constitute the Basis of School Mathematics?

Many people have wondered whether naturally occurring everyday situations will immerse students in learning situations diverse and consistently challenging enough to provide a wide-ranging background in mathematics. Their doubts are not without merit. Schools are in the very business of introducing students to novelty. This does not mean that they provide more satisfaction than work or that they are capturing the hearts and minds of most students. However, the routine day of school presents students with new problems to think about, while the routine day of work may or may not do so. Construction foremen typically have to grapple with wide ranges of problems dealing with measurement, visualization, estimation, and making adaptations to unique conditions. On the other hand elevator operators and assembly line workers may spend their whole workday without being challenged to solve new problems. As Smith's examples (in this volume) of mathematical activities in automobile production suggest, mathematics may be used in very different ways at the workplace. In some cases, tools and automatic processes may distance workers from using mathematical relations.

But we will overlook the most important contributions of life outside of school to mathematical learning if we restrict ourselves to the finished tools of mathematics: particular algorithms, material supports such as tables and graphs, notation systems and explicit mathematical terminology. Some of the most profound ideas in mathematics rest upon concepts learned in the physical and social world in what appear to be mathematics-free settings. Actions on physical objects—slicing modeling clay into several parts, joining multiple instances of elements, setting objects of one type in one-to-one correspondence with those of another type, nesting objects within others, dismantling toys—provide us with a wide range of experiences that later may prove crucial to understanding arithmetical and algebraic operations and relations among numbers, quantities, and variables. Commercial situations provide us with a wealth of knowledge about trading, profitability, interest, taxes, and so on that will prove necessary for understanding monetary mathematics and mathematical computation. The behavior of colliding objects, the exertion required to lift objects in different ways, judgments of the relative quickness of two automobiles, and experimentation with how our eyes work provide us with elaborate knowledge and intuitions about dynamics and statics, velocity, acceleration, and a host of other scientific concepts that ultimately play a major role in our making sense of advanced concepts in calculus, geometry, topology, and analysis.

We repeat: these situations do not provide finished knowledge. However they provide rich repertoires of experience, data, and schematized understandings of how things work without which advanced mathematical understanding would be inconceivable.

Everyday situations provide a foundation for constructing mathematical knowledge but not a rock solid one onto which students can quickly erect, with scaffolding supplied by teachers and parents, mathematical skyscrapers. When construction proceeds at a rapid pace, as it typically does, school mathematics will occasionally wobble on its intuitive foundations. For example, students may become puzzled when they discover that multiplying does not always make quantities grow bigger—a view long supported by their growing intuitions in elementary mathematics instruction. This fault can be superficially patched by telling the students that the old rules no longer apply and that rational numbers are different from integers. But a satisfactory fix of the problem requires examining the foundations and seeing how they can be accommodated to support the weight of new knowledge. For example, they may need to understand that fractions have both a multiplication- and division-like quality. The numerator of a fractional operator acts like a natural multiplier; the denominator acts like a natural divisor. Their relative magnitude determines whether the result will be greater, less than, or equal to the original quantity.

The construction site metaphor perhaps suggests that the upper floors will develop well once the foundations are solidly established. However, the relationship between intuition and new mathematical ideas is one of constant tension and readjustment. The Greeks of antiquity had to adjust their intuitions about number when they realized that the diagonal of a unit square could not be expressed as an integer ratio of the side. Similar tensions have arisen in the history of mathematics in the cases of Zeno's paradoxes (it takes a finite amount of time but an infinite number of steps to reach the tortoise), negative quantities (how can there be less than nothing?) and Cantor's infinities (how can one infinite set be greater than another?).

It is comforting to believe that everyday mathematics is reconcilable with the mathematics of mathematicians. But there are times these approaches will clash and it is instructive for us and for students to become aware of these mismatches. We laugh when we hear that the average family has, say, 2.3 children or that we need 7.3 buses to transport a certain number of people because we know that children and buses come in whole numbers. There is a sense in which even these "artificial" answers are true, and learning mathematics often requires temporarily suppressing common sense and traditional thinking in favor of following a stream of logic along its course.

Brenner (in this volume) notes a child who determines the relative cost of pizzas by comparing how much pizza she can buy with ten dollars in the two establishments. Her solution exemplifies the resourcefulness people use solving problems in everyday situations. But the teacher may not want to end discussion with this admittedly correct answer, particularly if she wishes to focus on the concepts of area and unit price (cost per square inch) that will later prove important in many mathematics and science contexts. Likewise, the boys who determine the cost of a turkey dinner by simply looking up the advertised price (see Guberman, 1999) have amusingly circumvented the purposes of the exercise. The turkey dinner task is in a sense a stage on which several parallel stories are being acted out. The students presumably know that the teacher does not simply want to know the cost of a meal. Instead, she wants them to partake in the planning of the meal, figuring out the nature and amounts of ingredients needed, configuring the quantities for the number of people invited, and then determining the cost per plate. Giving the teacher a predetermined price is like responding "yes" to the question, "Do you have the time?" Technically speaking, it is a valid answer.

It is not easy to say how much children should be left to their own devices in solving mathematics problems. Proponents of laissez-faire pedagogy would go to great lengths to favor student inventiveness over the appropriation of conventional knowledge. Some would go so far as to recommend that students create their own notational systems rather than be forced to adopt those created by others. The French approach to the didactics of mathematics (see Laborde, 1989) makes a strong case for a distinctly opposing view. Although they would encourage children to generate their own solutions and choices and recognize that mathematical knowledge grows around what are intensely personal activities, they are also concerned that children become skilled in using conventional representational tools.

The scope of everyday mathematics comes to a head in the chapters by Civil and Masingila in this volume. Masingila's overview of children's descriptions of mathematical activities out of school shows how their conceptions of mathematics affect how they see it happening in everyday settings and leaves us with a certain sense of discomfort regarding the scope of everyday mathematics. There seems to be relatively little mathematical activity in children's out-of-school activities, and when mathematics comes into play, it does not seem to call for a deep understanding of mathematical relations. This limitation also comes into mind when Civil questions whether one can build an entire mathematics curriculum around the everyday activities of immigrants. Cultural and social environments that support the construction of mathematical knowledge may constrain and limit the knowledge children and adults will come to develop (Petito & Ginsburg, 1982; Schliemann & Carraher, 1992; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998). Finally, as suggested by Civil's and Brenner's data, once transposed to the classroom setting the problem are no longer the same.

Activities in classrooms can be organized so that children will experience a wider range of situations and tools for using mathematical concepts and relations thus allowing them to explicitly focus on mathematical concepts from different perspectives. Schools can also engage children in using a variety of symbolic representations such as written symbols, diagrams, graphs, and explanations, which constitute opportunities to establish explicit links between situations and concepts that would otherwise remain unrelated. Such are the activities that will allow children to understand mathematical concepts as belonging to, in Vergnaud's (1990) terms, conceptual fields.

Conclusions

Is everyday mathematics really relevant to mathematics education? Yes, but not as directly as many have thought. The idea that we can improve mathematics education by transporting everyday activities directly to the classroom is simplistic. A buying-and-selling situation set up in a classroom is a stage on which a new drama unfolds, certainly one based on daily commercial transactions, but one that, as Burke (1945/1962) might have expressed it, has redefined the acts, settings, agents, tools, and purposes.

Classroom goals are different from but no less complex nor cultural than goals in out of school settings. New situations challenge students to go beyond their everyday experience, to refine their intuitive understanding, and to express it in new ways. In a school setting these situations are always to some extent contrived. When the contrivances lead to playful puzzle-solving inquisitiveness and debate, teachers are rightfully pleased. When they fail to engage students, the situations present themselves as artificial. Mathematics teachers cannot totally renounce the use of contrivance or, to use a less charged term, staging, because naturally occurring everyday situations are not sufficiently varied and provocative to capture the spectrum of mathematical inquiry. This leaves teachers with immensely difficult dramaturgical problems, particularly when the students are leery of book knowledge and unfamiliar notational systems.

The outstanding virtue of out of school situations lies not in realism but rather in meaningfulness. Mathematics can and must engage students in situations that are both realistic and unrealistic from the student's point of view. But meaningfulness would seem to merit a consistently high position on the pedagogical pedestal. One of the ways that everyday mathematics research has helped in this regard has been to document the variety of ways people represent and solve problems through self-invented means or through methods commonly used in special settings. By explicitly recognizing these alternative methods of conceiving and solving problems, teachers can hope to understand more clearly how students think and to appreciate the chasms they must sometimes cross to advance students' knowledge.

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Notes

[1] This peculiarity is closely related to the "learning paradox" (Bereiter, 1985). Bereiter was emphasizing the tension between "what one already knows" and "what one is capable of learning." Our stress is on the tension between "what one already knows" and "what others presume about our knowledge."

[2] The bulk of concept formation empirical work in the United States between 1915 and 1970 clearly adheres to the common element view of concepts. Piaget's work (and occasional treatises on "relational concepts") is much more consistent with Cassirer's idea of concepts based on function as opposed to substance (and common elements). However, the relation of universal structures to particular content continues to be a source of controversy in Piagetian circles (Schliemann, 1998). Many believe that Piaget presumed that knowledge about particular concepts would develop once universal structures were firmly in place. There is no question that he subordinated learning to development, making it seem, at times, that learning was an epiphenomenon of development rather than a process that required specific constraints to be overcome and new paths to be pursued, as we would argue.

[3] We wish to thank Romulo Lins for introducing us to the intriguing Goldbach conjecture many years ago.

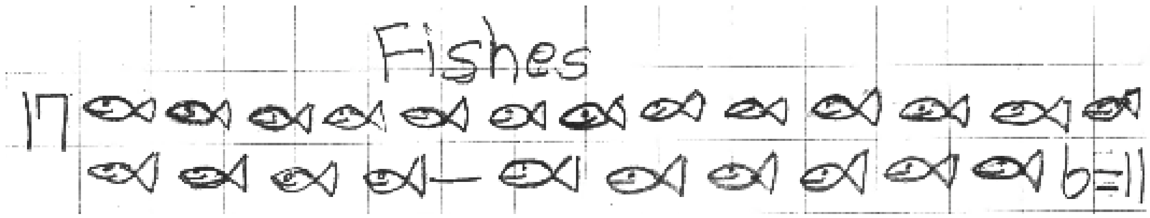

Figure 8.1 – One example of children's representation of 17 fish minus 6 dead fish.

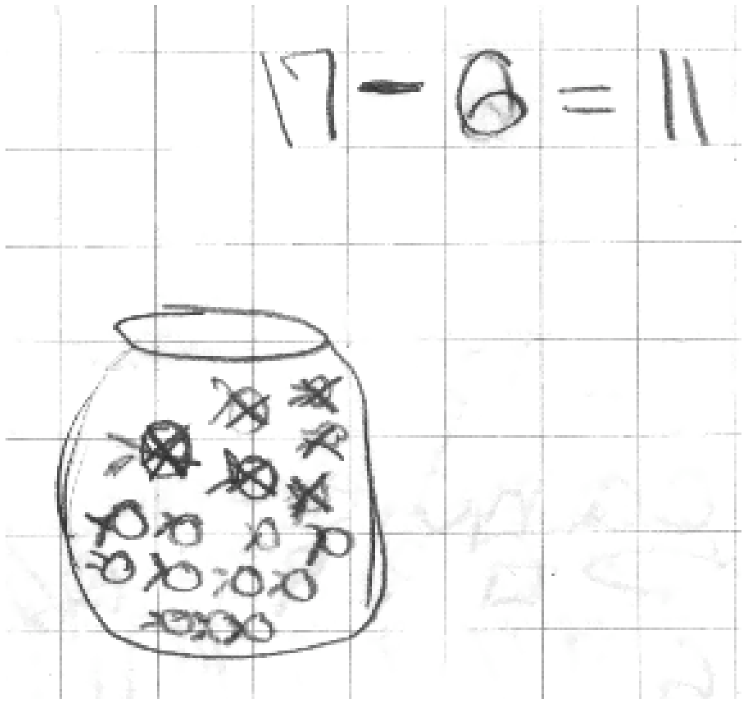

Figure 8.2 – Another example of children's representation of 17 fish minus 6 dead fish.

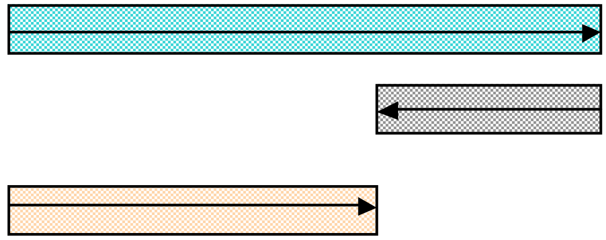

Figure 8.3 - Our vector representation proposed for the problem of seventeen fish, six of which died.