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The Evolution of Mathematical Reasoning: Everyday versus Idealized Understandings

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Developmental psychology lacks a theory of mathematical reasoning that accounts for how learners appropriate conventional symbol systems into their thinking. In this essay we attempt to consider how students' mathematical thinking evolves not only as a result of their actions and everyday experiences but also from their increasing reliance on introduced mathematical principles and representations. First we contrast how certain mathematical ideas are represented diversely in school and out of school. Then we exemplify, from our own research, how 8- to 10-year-old children's personal representations come to face with (what for them are novel and for us are conventional) representations involving algebraic concepts. Finally we explore some implications for theories of instruction and long-term development of mathematical reasoning.

Mathematical understanding is both a personal and a cultural enterprise. It is personal insofar as it entails invention and rediscovery even when people are merely learning facts and conventions. It is cultural because it relies on conventional symbol systems and social contexts. Children construct a foundation for logical and mathematical thinking on the basis of their direct experience and reflection. But early on they draw from their experience in social contexts. Conventional representations and reasoning practices profoundly affect the course and possibly the very nature of their mathematical thought. Any theory of development runs into the fact that mathematical concepts and representations have a history of their own.

Developmental psychologists have striven to identify presumedly universal, atemporal, and acultural characteristics of thinking and learning. In doing so they have tended to miss how development benefits from what has been

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collectively learned. Even though contemporary authors have begun to recognize the need to develop theories on the basis of individuals whose thought has been socialized almost from the moment of birth, much remains to be done. In the area of mathematical reasoning, we need to consider how particular external representational systems play a role in the evolution of thinking.

In this article we examine how students' mathematical thinking evolves from their previous understandings and experiences out of school and from their participation in school activities as they become acquainted with formal mathematical principles and conventional mathematical notation. We (a) consider developmental psychology's contributions to the study of mathematical reasoning; (b) contrast traditional mathematics instruction with informal mathematical practices and conceptions children develop on their own outside of formal schooling; and (c) provide examples from our classroom research in which formal instruction takes into account informal, everyday understandings. We argue that the understandings children develop outside of school are an essential prerequisite for any effective program of instruction. But there is more to mathematical understanding. Any analysis of how children develop mathematical knowledge must take into consideration the tools and representations they come to use and understand as they participate in formal mathematics instructional activities.

PSYCHOLOGY AND MATHEMATICAL LEARNING

Many issues of learning fall outside the purview of existing psychological theory, and as the child progresses, theories of development will have to cede ground to analyses of the particular mathematical issues learners confront (e.g., their notational peculiarities and their relations to mathematical content). What can we expect from psychological theory given that the development of mathematical understanding cannot be reduced to psychological processes?

Actions, Everyday Contexts, and Mathematical Reasoning

Piaget's proposal that *human action* and its internal representations provide a basis for thinking continues to offer a striking alternative to the empiricist premise that perception is the source of knowledge. In recent years this insight has inspired many discussions about the nature of mathematical knowledge, including the process–object tension (Sfard & Linchevsky, 1994) and the metaphorical thinking and body image schemes underlying mathematical concepts (Lakoff & Nuñez, 2000). It has influenced our own work on everyday mathematics (e.g., mathematical operations as grounded in the actions of buying and selling, as studied by Nunes, Schliemann, & Carraher, 1993) as well as our more recent work on children's early algebraic understanding—for example, the bodily displacements implicit in operations on number lines and in Cartesian graphs (Carraher, Brizuela, & Earnest, 2001; Schliemann, Goodrow, & Lara-Roth, 2001).

Even though Piaget acknowledged the contribution of sociocultural factors to the development of mathematical understanding, his theory almost exclusively emphasizes children's own actions and reflections upon these actions as the source of mathematical understanding. Piaget largely ignored how mathematical procedures, notation, and formal contents may contribute to the development of mathematical thinking.

In recent years developmental psychology (or at least some of its researchers and theoreticians) moved away from the search of general, invariant cognitive development mechanisms as they sought to clarify the contextual nature of cognition. Neisser (1976) recognizes the contextual nature of cognition in his critique of models of memory processes based solely in the results of laboratory studies. Bronfenbrenner (1979) calls for studies of how settings and the ecological environment mediate cognitive processes. Ceci (1990, 1993) proposes a contextual model of intelligence where the potential for intellectual achievement develops as a result of one's experience in specific contexts. Leontiev (1981), Luria (1976), Vygotsky (1978), and Cole (1988) emphasize the importance of social, historical, political, and economical changes for the organization and development of human cognition. Rogoff (1990) and Saxe (1991) attempt to reconcile the notion of individual cognitive development with sociocultural analysis and Lave (1988) argues for the dissolution of boundaries between the individual and the contexts where cognition takes place. It is now fairly widely accepted that specific contexts, far from being incidental, are essential to what is learned and thought.

Empirical studies show that cognitive performance may vary considerably across contexts. For example, children who fail in Piagetian conservation, class inclusion, or perspective-taking tasks demonstrate more advanced logical reasoning when interviewers ask the questions in slightly altered ways (see, among others, Donaldson, 1978; Light, Buckingham, & Robbins, 1979; McGarrigle & Donaldson, 1974). Children may also show different performance across contexts in multicausal reasoning (Ceci & Bronfenbrenner; see Ceci, 1990, 1993), syllogistic reasoning (Dias and Harris, 1988), and even microlevel cognitive strategies, such as the temporal calibration of one's psychological clock (Ceci & Bronfenbrenner, 1985).

We find similar results in the case of mathematical reasoning. What children seem to be able to do in the classroom or in formal interview or testing contexts may be rather different from what they may do in informal, out-of-school contexts (Carraher, Carraher, & Schliemann, 1985). The contexts, social goals, and values associated with activities appear to highlight different relations and evoke different approaches and representations. Children draw from the particular social and physical activities in which they engage (buying and selling, comparing, measuring, computing and solving mathematical problems in and out of school, and so on). Their ways of conceiving and doing mathematics owe much to the specific representations and tools they learn to use such as abacus, weights and measures, quantities they deal with, notational systems, and so on. (Nunes, Schliemann, & Carraher, 1993; Hatano, 1982).

There are several rich topics of mathematics where one can expect clashes between schemes that have evolved out of school and the structure of knowledge being introduced. Consider physical quantities for example. Mathematics, at least modern mathematics, has lots to say about numbers but surprisingly little to say about quantities. (Granted, when quantities are measured or counted they are associated with numbers.) This raises enough issues to keep psychological theorists busy for decades. Do negative quantities exist? And how are they related to negative numbers? If the concept of division stems from the social act of sharing, why is division by zero undefined? Why does it not result in infinity, since the quotient approaches infinity as the divisor approaches zero?

Thinking about actions on quantities constitutes an important basis for many mathematical concepts, including number. Number words and activities involving numbers and quantities pervade a child's environment from a very early age. Counting and sharing, buying and selling set foundations for future learning. The mathematics children come to use and understand in everyday situations may draw upon the same underlying properties they will learn about in school. However, these properties arise in very different systems of representation and in different motivational contexts (e.g., mental computation based on the structure of the monetary system used in everyday activities versus the written computation algorithms taught in schools). Eventually children will encounter tensions between their everyday experience and problems framed in mathematics lessons. Not all persons sharing a candy bar may insist upon receiving a full share; no person can divide a candy bar equally, no matter how hard they try. Some water remains in jar 1 when we pour the contents into jar 2, and some water even evaporates. If one repeatedly takes half of an amount of liquid, one eventually arrives at the point where there is no liquid left to pour. Over time students learn to suppress realistic concerns in order to focus on mathematical relations. The socialization of their thought to conform to mathematical convention is reminiscent of Luria's finding that formerly unschooled rural Asians exhibited syllogistic logic after several months of schooling (Luria, 1976).

Piagetian and sociocultural approaches to cognitive development provide crucial insights into the long-term development of children's understanding of basic logical and mathematical principles. Far less is known about how their understanding is reorganized through contact with mathematical symbol systems and tools (e.g., the conventions of the decimal system notation, fractional notation, and transformations across conventional measuring units). Mathematics may even involve principles that have no direct counterpart in everyday contexts and therefore require creative and metaphorical applications of schemes. Developmental psychology tends to examine how children develop logical and mathematical understanding, while educators are left to deal with learning. However if conventional symbolic representations indeed exert an influence over the direction and nature of reasoning this distribution of labor may prove inadequate.

Symbols, Conventions, and Mathematical Reasoning

Vygotsky's (1978) work has raised the intriguing prospect that representational tools, in mediating thought and communication, may actually transform cognition. One need not subscribe to the view that a representational tool immediately and directly influences cognitive processes. One could hold that tools channel and structure thought in ways that would not have otherwise occurred. Symbolic representations open up further avenues of thinking and evoke certain comparisons to things the learner already knows. The tools and representational systems used by the individual play a role in the structure and direction of mathematical thinking, allowing for different aspects of mathematical reasoning to come to the forefront. This is why schools have a legitimate interest in having students learn the ‘‘tools of the trade’’ of mathematics.

Although mathematicians (like students themselves) often create their own representations, and sometimes their own notations, very few prove themselves worthy of adoption by the mathematical community. As Cajori (1929/1993) notes, ‘‘many notations are invented, but few are chosen’’ (p. 337). He finds that no mathematician, with two possible exceptions, has ever introduced more than two notational conventions into the language of mathematics. Constructivist mathematics educators thus find themselves in a perplexing position: they want to encourage students to express their understandings through symbols meaningful to them. But they need to recognize that some representations offer more promise over the long run. A child's partially iconic drawing of a fishbowl with 17 fish, 6 of which are crossed out, may meaningfully convey the story that Mary had 17 fish, 6 of which died. But this representation may be ill-suited for other contexts and the notation ‘‘17 − 6’’ or displacements drawn on a number line will prove more generally useful: what the latter representations lose in reference to the particular problem context may be offset by their ability to represent a large set of possible contexts, with or without fish (Carraher & Schliemann, 2002). The zone of proximal development is less about the assimilation of knowledge and symbolic representations where there were none before than the gradual adjustment, progressive schematizing, and replacement of representations so as to become more coherent and applicable to a wide variety of circumstances.

Children may develop certain notions of probability before they have received instruction in mathematics. However, learning how to multiply opens up possibilities for approaching probability in systematic ways and to compare the relative likelihood of various sequences of events. It is true that children will not comprehend probability unless they have already acquired basic concepts such as ratio and proportion that require a long time to develop. These prerequisites themselves will have already benefited from a wide range of experience and exposure to conventional representations in natural language, diagrams, tables, and written notation.

A high school student of today can solve algebra problems that a Greek mathematician from antiquity would have been hard pressed to represent. Ancient Greeks represented problems essentially through natural language and geometry; modern algebraic notation emerged only in the past 500 years (Harper, 1987). Even when we take on such ‘‘simple’’ concepts as addition and multiplication, we find that students of today make use of representational advances such as place value notation and column multiplication that were practically unavailable to students before 1500.

Psychologists may see a ‘‘developmental pattern’’ in the emergence of concepts such as multiplication, ratio, and proportion—overlooking the fact that students in today’s schools are given explicit instruction about these topics from around 9 years of age. How are we to understand development when we leave the present historical moment? Are we to assume that European children in the middle ages who never learned to multiply, much less recognize the proportionality of equivalent ratios, did not fully develop? How do we handle the fact that Greek mathematicians of antiquity did not have a means to directly represent irrational numbers, yet a large portion of today’s high school students routinely operate on irrational numbers? Several years ago many Piagetian psychologists believed that proportional thinking was intimately tied to a period of formal operations. Now that we find that many preadolescent children comprehend ratio and proportion, what are we to conclude? Are these children entering formal operations earlier? Or was this characteristic unduly associated with the period? Is *any* mathematical content tied to invariant periods of cognitive development? What if we discover that, given the proper circumstances, young children can learn to make mathematical generalizations and use algebraic notation with meaning at the age of 9 years? Should we conclude that our theories of cognitive development were off the mark? Or that the children were going through ‘‘accelerated development’’? Should theories of psychological development define stages in terms of any mathematical content? If not, how can we distinguish stages of development? Or perhaps we need to abandon the goal of casting mathematical competence into the mold of development. Perhaps we should refer to the ‘‘evolution’’ of mathematical ideas to convey the idea that there is no single path of mathematical learning. These are not easy questions to answer.

When psychologists evaluate the ‘‘development’’ of children who have already entered school, they are not dealing directly with cognitive universals. Developmental psychology should not restrict itself to explaining learning in terms of universals but should identify issues, processes, and structures that place learning in a relatively long-term perspective. In the case of mathematical learning this can only be achieved if psychologists are willing to delve deeply into the intricate and subtle issues posed by the subject matter itself.

Beyond Developmental Psychology

Whenever one considers a particular mathematical topic, particular issues arise that cannot be foreseen by psychological theory. Division by zero becomes an issue in a Hindu–Arabic numeral system but not in the case of Roman numerals, for example. Several of the difficulties students encounter are associated with the paradoxes that the premise, ‘‘zero is a number,’’ brings to the fore—limits, infinitesimal quantities, and so on. In this regard, the history of mathematics may provide important insights into some of the issues students of today face.

Developmental psychology needs to address the specific issues involved in mathematical learning and evolution while considering the history of mathematics and the psychology of mathematics education. This relatively new field of study offers concepts such as *additive* and *multiplicative structures* that help us make connections across topics that would otherwise be difficult to make. It attempts to clarify how operations rely on generalized schemes that may be grounded in a variety of mundane situations. While bringing to the fore the role of quantities and quantitative relations in the evolution of mathematical ideas, it considers the tension between the logic of mathematics and the logic of everyday reasoning. It documents the evolution of students’ mathematical representations as they participate in out of school activities and in classroom instruction. Developmental psychologists have eagerly pointed out that what one learns depends on the learner’s level of development. They have been far less likely to note how development benefits from what has been learned and how cognitive functioning relies heavily on the tools one becomes acquainted with in schools.

In attempting to fully understand the development of mathematical reasoning we need analyses of how children learn as they participate in cultural practices, as they interact with teachers and peers in the classroom, as they become familiar with mathematical symbols and tools, and as they deal with mathematics across a variety of situations. The teaching and learning of mathematics as a discipline unfolds from children’s basic logical and mathematical understandings, leading to more general, complex, and explicit knowledge. To acknowledge this, however, is not enough. We need to analyze how children’s logical and mathematical understandings developed outside of schools can be further expanded as they participate in instructional activities. Ultimately we need to find ‘‘the most adequate methods for bridging the transition between (. . .) natural but nonreflective structures to conscious reflection upon such structures and to a theoretical formulation of them’’ (Piaget, 1970, p. 47).

In this article we first report on studies of mathematical understanding developed outside of school. We will contrast everyday mathematics to school mathematics, analyze the strengths and limitations inherent to its contextualized nature, and consider its relevance for the development of mathematical understandings in schools. We then show examples from our third-grade classroom intervention studies on early algebra that illustrate how children's mathematical understandings, while benefiting from everyday experiences, are extended as they have access to new experiences and new ways to represent quantities, events, and relationships.

EVERYDAY MATHEMATICS

Children and adults gain an important grounding in mathematical concepts through activities such as buying and selling, carpentry, weaving, lottery, agriculture, tailoring, and so on (see Nunes, Schliemann, & Carraher, 1993, and reviews by Carraher, 1991, and Schliemann, Carraher, & Ceci, 1997). People with restricted school experience can learn about arithmetical operations, the properties of the decimal system, proportionality, measurement, geometry, and probability.

In our earlier work (Carraher, Carraher, & Schliemann, 1982, 1985; Nunes, Schliemann, & Carraher, 1993) we interviewed young Brazilian street vendors as they worked selling goods and found that in 98% of the cases they gave correct answers to arithmetic problems involving prices of goods. When, a week later, in a school-like context, we gave them what we considered to be equivalent problems, the percentage of correct answers dropped to 74% in a verbal problem condition and to 37% in computation exercises that involved ''pure numbers.'' At work street sellers solved the problems using mental computation strategies that were not learned at school. In the school-like situation they attempted to use written school algorithms.

Motivation and social interaction might have accounted for the above differences in performance. To rule out these factors we conducted a study with 16 third-graders who, although not regularly working as street vendors, had experience dealing with money transactions (Carraher, Carraher, & Schliemann, 1987; Nunes, Schliemann, & Carraher, 1993). Each child was asked to solve 10 arithmetic problems embedded in the following conditions: (a) a simulated store condition where the child posed as a shopkeeper and the interviewer as a customer, (b) verbal problems, and (c) computation exercises. We presented the problems orally but the children had paper and pencil available in case they wanted to use them. The children's performance was significantly lower in the computation exercises and, when they chose to solve the problems orally, they were significantly more likely to succeed than when they relied on written computation algorithms. The following example (Nunes, Schliemann, & Carraher, 1993, p. 46) illustrates the differences between school and nonschool strategies in handling the problem ''200 − 35.''

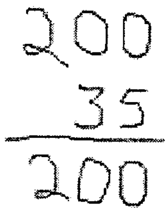


FIG. 1. A first attempt to compute 200 minus 35.

Ronaldo (R) begins by writing 200 and 35 (see Fig. 1) in accordance with the school algorithm. He writes 200 as the result, writing the digits from right to left.

> *R:* Five to get to zero, nothing. Three to get to zero, nothing. Two, take away nothing two.
> The interviewer (I) then asks: Is it right?
> *R:* No. So you buy something from me and it costs thirty-five, you pay with a two-hundred-cruzeiros note and I give it back to you?
> *I:* Do it again, then.

Ronaldo writes 200 and 35 in column alignment once more and proceeds as follows, this time getting 235 as a result.

> *R:* Five take away nothing, five. Three take away zero, three. Two, take away nothing, two. Wrong again.
> *I:* Why is it wrong again?
> *R:* Now you buy something and it costs thirty-five. You give me two hundred and I give you two hundred and thirty-five on top?
> *I:* Do you know what the result is?
> *R:* If it were to cost thirty, then I'd give you one hundred seventy.
> *I:* But it is thirty-five. Are you giving me a discount?
> *R:* One hundred sixty-five.

We might represent Ronaldo's final answer as follows:

$$35 = 30 + 5$$
$$200 - 30 = 170$$
$$170 - 5 = 165.$$

Although the street vendors did not explicitly express the associative nature of addition they revealed their implicit use of the property through the transformations they made on the values given. The standard school algorithm for column subtraction invokes the same general property but decomposes the given values in a different manner.

The street sellers develop a basic understanding about the properties of our numerical system. Their failure in school arithmetic arises not through cognitive deficits, but rather from troubles in adopting written symbolic systems and procedures. Computation algorithms provide students with symbolic representations and procedures that are not always understood. Moreover, algorithms are frequently used without reference to physical quantities or to the situation being dealt with. In contrast, the mental computation strategies developed as tools to solve problems at work reveal understanding and constant attention to the meaning of the situation and the adequacy of solutions.

Our studies suggest that everyday mathematics can provide a meaningful basis for the development of more advanced mathematical activities in school and to the meaningful learning of conventional symbolic systems. But, as is shown next, everyday experiences may constrain and limit the knowledge children and adults will come to develop (Carraher & Schliemann, 2002; Schliemann & Carraher, 1992; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998; Schliemann, 1995).

Generalization and Context Specificity

Schliemann and Acioly (1989) examined the generality of everyday mathematics by asking Brazilian lottery bookies to use what they knew about permutations of numbers, an important memorized feature in their work, to deal with problems involving permutation of letters. To a mathematician the problems are *isomorphic* in the sense that one can directly match the structure of one version to that of the other. However, some of the bookies viewed the problems with letters as completely different from problems with numbers, as the following transcript (from Schliemann & Acioly, 1989, p. 206) shows:

> *Interviewer:* I want you to find out in how many different ways you can arrange the letters in the word CASA (shows word written on paper) without leaving any letters out and without using any other letters.
> *Subject* (who has just given the permutations for a set of numbers): This one is even harder (than with numbers) because I can't read.
> *Interviewer:* But you don't have to read. I want you to tell me about how many different ways you can change the position of these letters.
> *Subject:* I can't do this.
> *Interviewer:* What if you try to do it as in the Animal's Game?
> *Subject:* This is very hard because reading is more difficult than working with numbers. I know how to do a few calculations but I don't know how to read.
> *Interviewer:* What if you make believe that ''C'' is a number like ''1,'' the ''A'' a number like ''2,'' the ''S'' is number ''3,'' and this ''A'' is number ''2.'' Couldn't you do it?
> *Subject:* No, because one thing is different from the other.

When we examined how the bookies' responses related to their school experience, we found that only those with less than 1 year of schooling responded incorrectly to the letter version of the problems. Even very short exposure to schooling can make a difference in people's reasoning (see Luria, 1976; Scribner, 1977).

Our next step was to clarify whether procedures that were not solely based on memorization could be seen as applicable to other contexts. We did this by examining the generality of everyday price computation strategies in a study among fishermen in a community in northeastern Brazil (Schliemann & Nunes, 1990). Fishermen's strategies to compute the price of many items based on unit price appear to go beyond use of a memorized procedure since they are able to invert the strategies in order to compute the price of one item given the price of more than one. We found that even those fishermen who had never been to school used the computation procedures to solve problems relating amounts of processed to unprocessed seafood, something they never had to do in their work.

In another study (Schliemann & Magalhães, 1990) we asked female cooks enrolled in an adult literacy class to solve missing value proportionality problems in three contexts presented to them in three different orders. Two of the contexts (sales transactions and cooking recipes) were part of their everyday experience. The third was an unknown context of a mixture of pharmaceutical ingredients. Cooks who first solved price problems nearly always worked out precise answers to these problems. By contrast, the cooks who responded first in the context of pharmaceutical and recipe problems tended to give nonproportional, and hence incorrect, answers. In this case, about half of the solutions for recipes were estimates. For medicine problems, roughly one-half of the answers appeared to be guesses or the result of meaningless computation. Results for recipe problems in a second presentation, after solution of sale transaction problems, were clearly better (61% as opposed to 18% correct responses). The percentage of correct answers for medicine problems given after sale problems and before recipe problems, however, remained low (27%).

These results suggest that everyday situations can ''prime'' reasoning in novel contexts. The fisherman had experience with ratios between quantities of processed versus unprocessed seafood and the cooks had experience with quantities involved in recipes. Those experiences seem to allow them to recognize that problems involving such relations can be solved through the same computation procedures they use for prices. But when subjects had no prior experience with the context, as was the case of medicine formula in the cooks study, they did not assume that the amounts in the problem should involve the same relationships.

Problem solving is initially tied to the meanings and goals of particular situations. With increasing practice and experience, people begin to develop an understanding of proportional relations as a scheme they can make use of in other contexts. This transition fits Piaget and Garcia's (1991) view that logical and mathematical knowledge have foundations in a logic of meanings (Schliemann, 1998). The logical properties first appear in specific contexts and are linked to the properties of the objects and situations at hand. But with increasing activity and practice, general and systemic logical understanding may develop. Different cognitive and sociocultural factors, however, such as the value attributed to everyday practices, the conceptual understanding underlying everyday solution strategies, or how much is known about how variables in different contexts interrelate, are crucial components in this process.

Everyday Proportional Reasoning and Mathematical Principles

Street sellers' mathematics is intimately tied to the meaning and goals of the situation at hand, with numbers being always used in connection to their physical referents. Street sellers typically start from the price of one item and perform successive additions until they reach the number of items to be sold (Carraher, Carraher, & Schliemann, 1985; Nunes, Schliemann, & Carraher, 1993; Schliemann & Carraher, 1992). Vergnaud (1983) describes this strategy as a ''scalar approach''; it is sometimes referred to as ''building up.'' If we try to understand their procedure in terms of displacements in a function table, they work as if moving down the number column and the price column, summing money with money, items with items. In contrast, the functional approach focuses on the ratio between any two values in a row. This latter approach arose later historically and is more difficult for students to understand, but nonetheless has the advantage of being more closely tied than the scalar approach to functions and their algebraic description.

The following is an example of a coconut seller's use of the scalar strategy to determine the price of 10 coconuts at 35 cruzeiros each: ''Three are one hundred and five, with three more, two hundred and ten (pause). There are still four. It is (pause) three hundred and fifteen (pause), it seems it is three hundred and fifty'' (Carraher, Carraher, & Schliemann, 1985).

The street sellers' scalar approach involves a linking of a unique y value to each value of x and, as such, captures the essential idea of a function and reveals an implicit understanding of proportionality. It may therefore constitute a meaningful initial approach to solve multiplication and proportion problems. But this understanding may be limited to mathematical principles that are relevant to the specific goals of the situation while principles that are not relevant to these goals are never considered. The commutative property of multiplication as applied to repeated additions seems to be a case in point.

We asked Brazilian school children and street sellers who had received little or no instruction on multiplication to solve aloud pairs of verbal problems where they had to compute the price of a certain amount of chocolates based on unit prices (Schliemann, Araujo, Cassundé, Macedo, and Nicéas, 1998). The following is an example of the problems pairs we used:

> Type 1: A boy wants to buy chocolates. Each chocolate costs 50 cruzeiros. He wants to buy 3 chocolates. How much money does he need?

> Type 2: Another boy wants to buy a type of chocolate that costs 3 cruzeiros each. He wants to buy 50 chocolates. How much money does he need?

Participants first solved a problem where the larger number denoted the price of one item and the smaller one indicated the number of items to be bought. Immediately after they were given the corresponding problem where the smaller number denoted price and the larger one denoted number of items and were asked whether they knew its answer without doing any computation. If they used the former problem to answer the latter, we took this as an indication that they relied on the commutative property of multiplication. The group of school children who had received school instruction in multiplication (second- and third-graders) solved the first problems in each pair via multiplication and frequently relied on the commutative property to answer the second problems. In contrast, street sellers tended to use repeated additions throughout and rarely invoked the commutative property to answer the second problem. Instead, they successively added the number of cruzeiros, a cumbersome procedure leading to frequent errors if they had to add, for instance, 3 cruzeiros 50 times.

The above results suggest that, although people can learn meaningful mathematical ideas in mundane, nonacademic situations, they nonetheless need access to new symbolic systems and representations they are not likely to acquire out of school. In what follows we look at ways we have tried to build on children's understanding while introducing more novel representations. The mathematical content concerns linear functions.

LINEAR FUNCTIONS, TABLES, AND GRAPHS IN THE CLASSROOM

Linear functions are often approached through equations such as ''$y = mx + b$,'' where x and y are (independent and dependent) variables, m is a constant of proportionality, and b is the y intercept when the function is graphed as a straight line in a Cartesian space. Functions and rates can be represented in many additional ways such as tables, graphs, and fractions. Students begin understanding (linear) functions and (constant) rates long before they make any sense of an expression such as $y = mx + b$. Educators effectively teach about functions and rates long before showing such expressions to students. For instance, a multiplication table might be thought of as an embodiment of the expression $y = mx$, where x and y are integers along the margins and m corresponds to the number in the m times table.

In a pilot study with a third-grade classroom (Schliemann, Carraher, & Brizuela, 2000, 2001; Carraher, Schliemann, & Brizuela, 2000) we attempted to introduce linear functions. When we gave them an incomplete data table with number of items in one column and the corresponding prices in the other, we found that the students could correctly fill in the tables, but they did so in a columnwise fashion as if they were solving two unrelated problems. Their approach was generally consistent with scalar reasoning that flourishes out of school settings. But because we wanted them to eventually understand expressions such as $y = 3x$, we needed to discourage students from using their ‘‘double sequence’’ approach to tables.

First we broke the order of items in the table so that the columns, when read downward, no longer expressed easily interpretable number sequences. This puzzled the students and most still tried to use the standard building-up strategy. We then moved to a ‘‘Guess My Rule’’ game to eliminate altogether the possibility that students would approach the problems in a ‘‘downward’’ fashion. For each step in a game we presented an input value and asked students to predict an output value. This corresponded generally to the idea of filling out a table, row by row. Given the rowlike nature of the problem and the fact that we followed no apparent order when moving from one input to the next, there was no way for students to guess the output values by scanning the values in a downward fashion. This proved to be a useful way to encourage children to focus on the functional relationship. Eventually the children began to adopt mapping notation (e.g., $n \rightarrow n + 3$) we introduced to summarize the rules. The arrow was read as ‘‘becomes.’’ With the mapping notation they were able to solve problems corresponding to linear functions (e.g., $n \rightarrow 2n - 1$). Such notational conventions lend themselves not only to the problems at hand. We have found that 9-year-olds increasingly use similar notation to express relations among unknown values in other contexts (Carraher, Schliemann, & Brizuela, 2001).

The introduction of ‘‘*n*’’ to represent any value appeared to help children move from the computational aspects of the task to generalizations about how two sets of values were interrelated. However, on some occasions children were inclined to instantiate variables—to assign *fixed* values to what were meant to be *variable* quantities—without recognizing their general character.

The following interview with two of the children exemplifies the tension between instantiation versus generalization (Carraher, Brizuela, & Schliemann, 2000). The context was a height problem in which Martha was said to be 3 inches taller than Alan, but no information about Martha’s or about Alan’s height was provided. When the interviewer suggests calling Alan’s height x, one of the children believes that it would be strange to do so:

> *David:* . . . Do you think it’s strange (to call Alan’s height *x*)?
> *Jennifer:* Yes, ’cause it has to, it has to have a number. ’Cause . . . Everybody in the world has a height.

Jennifer, 9 years, believes that it would be inappropriate to use the letter x, representing *any* height, to describe Alan, since Alan must have a particular height. The interviewer turns the discussion to another context to see if this will eliminate her concern:

David: OK, I'll tell you what: I'll take out, I'll take out a nickel here, OK. And I'll give that to you for now. I've got some money in here [in a wallet] can we call that *x?* (hmmm.) Because, *whatever it is,* it's that, it's the amount of money that I have.
Jennifer: You can't call it *x* because it has . . . if it has some money in there, you can't just call it *x* because you have to count how many money [is] in there.
David: But what if you don't know?
Jennifer: You open it and count it.

Jennifer insists that it would be improper to refer to the money in the wallet as *x*, since it holds a particular amount of money. The remarkable thing here is that Jennifer has learned about the concept of mathematical variable according to which letters stand not for single unknown values but for a whole set of values (input and output sets, for instance). Her discomfort stems from the fact that the interviewer's example does not conform to her expectation—encouraged by activities such as the ''Guess My Rule'' game—that letters be able to stand for multiple values.

In fact, it is somewhat curious that a variable stands for many values, yet we exemplify or instantiate it by using an example for which only one value could hold (at a time). Jennifer eventually reduces her conflict by treating the amount of money in the wallet as, hypothetically, able to take on more than one value: ''The amount of money in there is . . . *any* money in there. And after . . . if you like add five, if it was like . . . imagine if it was fifty cents, add five more and it would be fifty-five cents.'' Fifty cents is only one of many amounts that the wallet *could have* (in principle) held.

When the conventions regarding variables clash with Jennifer's understanding of everyday situations (people have one height only; purses contain a particular amount of money), Jennifer actually reconceptualizes the everyday situation to conform to the mathematical conventions. It is precisely this sort of example that compels us to conclude that mathematical conventions can exert an influence over reasoning.

Creating Contexts for the Graphical Representation of Functions

In an ongoing longitudinal study, we explored how 48 children in three third-grade classrooms deal with linear functions as they start being introduced to the conventions of graph representation.

Graphs constitute symbolic conventional systems for representing certain quantitative aspects of changes in quantities and events. The conventions adopted in a graph obey a coherent set of rules so that the spatial relationships in the graph must be consistent with the relationships between the quantities or events they supposedly describe. In the case of graphs of functions a single line can represent the infinite number of possible number pairs that satisfy a certain function. Function graphs also allow one to represent and to visually compare multiple functions. As such, the graph of a function captures, differently than a function table, the essence of a functional relationship.

Piaget's extensive work on children's conception of space (Piaget & Inhelder, 1948/1956) and geometry (Piaget, Inhelder, & Szeminska, 1948/1960) shows that children as young as 9 or 10 years of age show an understanding of vertical and horizontal dimensions as a coordinate system. Thus we could assume that most of our third-graders would have some understanding about the coordination of vertical and horizontal lines in space. Our challenge was to create situations that would provide the ground for children to focus on functional relationships as they learned the conventions for graphing functions. As is shown, this involves constant attention to reconciling everyday experiences and mathematical relations.

We had been working with these children since they were in second grade as part of a longitudinal study aimed at exploring the algebraic character of arithmetic. The intervention consisted of eight 90-min weekly classes given during their second grade. During their first term in third grade, in eight weekly meetings, we explored additive functions (Carraher, Brizuela, & Earnest, 2001). In the second term of Grade 3 we focused on multiplicative functions and graphical representation (Schliemann, Goodrow, & Lara-Roth, 2001). On the first day of multiplicative functions activities we presented the following statement to the children: ‘‘Karen has twice as many dollars as Franklin.’’ We would be treating this statement as a function that can be solved by many pairs of values. As such, we were departing from its normal everyday interpretation as a statement about the relation between two particular values. This shift in focus from relations between particular number to sets of numbers (or variables) is fundamental to the early algebra program we were exploring.

Children immediately started giving examples of how many dollars Karen and Franklin could have, consistent with the ratio of 2 to 1. The instructor listed the value pairs in a two-column table, suggesting new values for Franklin’s column while the children computed the corresponding value for Karen. The children concluded that ‘‘twice as much’’ meant the same as ‘‘double.’’ When asked to complete the statement ‘‘Franklin has (blank) as many as Karen,’’ some children suggested that the statement should become ‘‘Franklin has twice as less as Karen’’ and the instructors then introduced the expression ‘‘half as many.’’

Each child then received a handout with the statement ‘‘Karen has twice as many dollars as Franklin’’ with the instruction to ‘‘Show this in as many ways as you can.’’ Children worked in groups and the researchers joined the different groups discussing the work. The most common representation consisted in attributing a number value to Karen’s amount of dollars, either by multiplying it by 2 or by adding it to itself. A few children, spontaneously or after prompt, made use of the algebraic representation for variables that had been introduced a few weeks before, representing Franklin’s amount as N and Karen’s as $2N$ or Karen’s amount as N and Franklin’s as $^1/_2N$.

We then took the children to the gym and had them make two parallel number lines, with values labeled from zero to 12 spaced about $1^1/_2$ feet apart on the floor of the gym. In a series of trials we asked one child to choose a value for Franklin in one line and then invited a second child to move in the other line to the number position that represented Karen's amount, always maintaining the relationship ''twice as many.'' The rest of the class sat in the bleachers watching and occasionally recommending where each child should go.

After a few trials, we rotated Karen's line at the origin by 90°, whereby it became a *y* axis, perpendicular to Franklin's line (the *x* axis). After a few trials with a pair of children moving along the line and discussions about how the number taken by a child was related to the number for the other child in terms of doubles and halves, we helped the children in the class ''plot themselves,'' one by one, at the intersections of invisible projection lines in the graphing space. For example, if Franklin's amount was $5.00, the child would plot herself at the intersection corresponding to (5, 10).

Some children were initially uncertain about where they should stand and needed help from the instructor or from other children. Others initially walked diagonally from the number on one line directly to the number on the other line. With prompting they then moved to the imagined intersection. The positions on the ''graph'' were named (0,0), (1,2), (2,4), (3,6), and so on, and the child in a position received a piece of paper with his/her ordered pair written on it. Later, we gave all the children located on the ''double line'' a string to connect their points. We then proceeded in a similar fashion for the case of ''three times as many.'' The class ended with two plotted lines of children, the *doubles* and the *triples.*

The following week we introduced a gridlike diagram to depict the graph made in the gym. The children were quick to note that the implicit perspective was that of a person located high up above the gym floor, looking down. We displayed the statement ''Karen has twice as many dollars as Franklin'' and asked the children to draw a table of possible values of Karen's and Franklin's money and to find the places in the graph corresponding to each possibility. Throughout the discussion that followed the instructor and the children referred to the positions specific children took in the larger scale graph. We then worked with another statement: ''Ann has three times as many dollars as Franklin.'' We were surprised by how easily most children in the three classrooms could find the intersection points and could say what the ordered pair at each point represented. Drawing the function line, however, was still an issue that not all of them seemed to have grasped. Several children connected the points from diverse functions as they might in a connect-the-dots drawing.

In the following week we introduced a new context for Cartesian graphs. We asked the students to imagine the intersections in the graph space as places where a table and chairs would be placed in the gym. Each table would have a certain number of chairs where children could sit. On each table there would be a certain number of candy bars to be shared equally by the children sitting at the table. We began by asking the students to locate the points corresponding to a table with four chairs and four candy bars and then a table with six chairs and six candy bars. The students could easily do so and, after some discussion, they agreed that if the candy bars were equally shared at each table, children at the ‘‘four for four’’ table each get as much candy as the children at the ‘‘six for six’’ table. Albert expressed this as ‘‘It wouldn't matter because if he [the child] went to [the table with six candy bars and] six people, he would, you would get one candy bar.’’ The instructor asked whether there would be a better table to sit at, where one would get more candy than in the first two. Jessie proposes that the table with five chairs and six candy bars would be better, and he locates this case at the appropriate intersection on the graph. Erika immediately proposes: ‘‘I'd rather go to the seven-to-one table’’ and correctly locates the coordinate (1, 7) corresponding to the table with one chair and seven candy bars. The instructor then asks the students to find the worst table: the table to send someone who misbehaved. Eric chooses the point corresponding to the table with seven chairs and only one candy bar. As children discussed different points in the graphing space, the instructor also showed and asked the children to represent the ordered pairs corresponding to the tables as fractions.

Throughout the discussion the children repeatedly drew upon their prior experience in the gym and their everyday experience with sharing candy bars and fairness to lend meaning to the points in the graph and to the written representation of ratios as fractions. As the class continued, Paul proposed that the candy bar on the table with seven chairs could be a Hershey candy bar that comes with squares, easy to break apart. He drew a candy bar with two lines by seven columns of squares, explaining that each person would get two of the small pieces he drew.

Everyday referents helped the children but also led to choices that were not consistent with the mathematical model under discussion. For instance, as the class compared the points representing tables on the graph grid, Paul considered that the four-to-four table was a better choice than the one-to-one table because ‘‘Someone may not want their candy bar.’’ Another child proposed that in the table with two chairs and four candy bars not everyone will get two candy bars because one person could get one and the other could get three candy bars. In these occasions, the instructor appealed to the idea of fairness in sharing the candies so that each child sitting at one table would have the same amount as the others. The children adopted this constraint in solving the problem in Fig. 2. With this problem we hoped to start helping them to look at the lines in the graphic space as representations of distinct functional relationships.

The children worked in groups while the instructor and other members of the research team circulated around the class, offering help when needed. After answering the questions, the children located the points corresponding to each table on a graph where the *x* axis denoted the number of chairs or

Table A has 3 candy bars and 2 chairs.
Table B has 6 candy bars and 4 chairs.
Table C has 5 candy bars and 2 chairs.

1. Show how you would divide the candy bars equally among those at each table.

2. Which table would you sit at? Why? Convince us that you chose the best table.

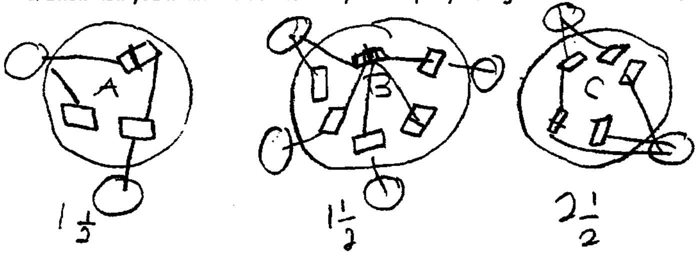
Are any tables the same?

FIG. 2. Finding and comparing ratios.

people sitting at each table and the *y* axis referred to the number of candy bars on the tables. The children used interesting sharing strategies to distribute the candy bars, most of them concluding that those sitting at tables A and B would each get one-and-a-half bars and those at table C were better off since they would get two-and-a-half bars. Most of them also located the points corresponding to each table on the graph. Some children correctly plotted the lines corresponding to different ratios and explained why they did so. Jennifer (not the same girl referred to above) found the answers to the questions using the diagrams in Fig. 3.

Table A has 3 candy bars and 2 chairs.
Table B has 6 candy bars and 4 chairs.
Table C has 5 candy bars and 2 chairs.

1. Show how you would divide the candy bars equally among those at each table.



2. Which table would you sit at? Why? Convince us that you chose the best table.

Are any tables the same?

C because has more candy.

The tables the same are AB.

FIG. 3. Jennifer's work to determine the best table.

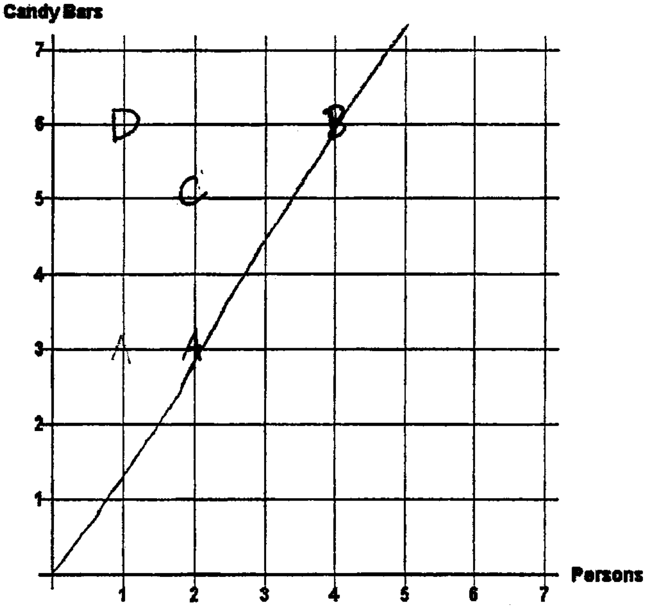


FIG. 4. Jennifer's graph. A, B, and C refer to the tables in the problem.

She then located the tables on the grid. She had first mistakenly placed table A at (1, 3), but then correctly placed tables A, B, C, and an extra table, D, with one chair and five candy bars (see Fig. 4).

Before she drew the line linking the points representing tables A and B and the origin, Darrell asked her the following:

Darrell: . . . What is the difference between tables A and B?
Jennifer: Nothing.
Darrell: Nothing? Are they the same thing?
Jennifer: Yeah.
Darrell: Oh. So do they fall in the same points in the graph, since they're the same thing?
Jennifer: No.
Darrell: How come?
Jennifer: Because, they're different but they're the same.
Darrell: Well, is there any way that we can show on the graph, that they're the same? Even though they're on different points?
Jennifer: [Looks at the graph for a few seconds.]
Darrell: Let's think about it this way. Do you remember we were talking about, if there's one per—the one, if there's one person and one candy bar, it's right here, there's two people and two candy bars, it's right here, three and three, and four and four, and five and five. And then we connected them?
Jennifer (nodding): Oh! They go on the same line.
Darrell: Why do you think they go in the same line?
Jennifer: Because they're the same.

Darrell: So, can you show me where that line would be?
Jennifer: [Draws the line starting at the origin, moving through points A and B.]
Darrell: Can I ask you a question? How come you didn't draw it like this—go up to A and then over to C, then over to D and then over to B?
Jennifer: Because they're not the same.
Darrell: They're not the same. Ah. So, which points will fall on this line?
Jennifer: The points that are the same.

The concept of equivalence may rest upon a logical structure but it goes through a socialization. There are many senses in which the points (6, 4) and (3, 2) are *different.* But Jennifer is learning that what counts as being the *same* (for the teacher who introduced the conventions) is whether the amount of candy per person is equal in each case. The straight line visually depicts those points for which the amounts per person are equal.

In this class children built upon their previous understandings of space and the specific experience with the large-scale graph in the gym. They also drew upon understandings about sharing fairly. Throughout this process the tension between everyday understandings and mathematics structures was constantly present, sometimes helping, sometimes hindering the analysis of pure mathematical relations. As the instructor explicitly acknowledged this tension the students were able to focus upon the mathematical relations, hopefully being introduced to some of the crucial issues involved in the interplay between mathematical models and everyday contexts.

DISCUSSION

Children build a foundation for logical and mathematical thinking from their actions and reflections. Logical and mathematical thinking further evolves as children engage in social interactions, games, commercial transactions, and discussions with others. As students they encounter conventional representations and reasoning practices that will affect the course and even the nature of their mathematical thought. A theoretical account of mathematical reasoning requires uniting the findings of developmental psychology, everyday mathematics, and mathematical learning in schools. It will also require a careful analysis of the structure and semiotics of mathematics itself.

Developmental psychology provides important contributions to the study of mathematical reasoning among children. Cognitive development theory roots knowledge in actions and schemes from infancy and early childhood—schemes that later prove fundamental even to advanced mathematical thinking. Studies of everyday mathematics draw attention to how a variety of commercial and measurement contexts play a role in the shaping of mathematical thought. Buying and selling, for example, provide motives and logic that help render computations about quantities meaningful. But this is not all there is to mathematical learning and reasoning. Schools introduce students to a wide variety of tools and representations they would not have invented on their own nor understood without dedicating themselves to the task of understanding them.

Insofar as tables of values entail functions, one can look to studies of children's understanding of functions from developmental psychology and thereby relate the present task to issues such as dependency among variable actions, inverse actions and operations, and so forth. When, as in several examples we presented here, the social context concerns the buying and selling of items, we can relate the students' approaches to the ''building up'' strategies of street vendors. We will not be surprised by the fact that, given the choice, children prefer to maintain separate operations on quantities of like nature—numbers of items on numbers of items, prices onto prices. We can also relate this to the historical fact that ratios of different quantities emerged only recently in Western mathematics. But schooling is not merely about watching how development runs its course. Schools have an agenda. The fact that students will eventually have to understand and use algebraic expressions such as ''$y = 3x + 6$'' has significance for how one deals with tables when children are 9 or 10 years of age. Children's strategies are a nice start, but they approach functions in ways that obscure the relation to representations via an equation. (Think of the distance between an expression such as ''for every 2 apples Mary has, John has 3, and the equation, $y = 2/3\ x$''.) To teach effectively, an educator needs to constantly evaluate the conflict or fit between what children bring to the learning situation and where learning is headed—in large part due to the organization of the curriculum.

The design of classroom activities requires (a) considering children's previous understanding and intuitive ways of making sense and representing relationships between physical quantities and between mathematical objects; (b) providing opportunities for children to participate in novel activities that will allow them to explore and to represent mathematical relations they would otherwise not encounter in everyday environments; (c) exploring multiple, conventional, and nonconventional ways to represent mathematical relations; and (d) constantly exploring the matches and mismatches between rich contexts and the mathematical structures being dealt with.

Our study of the graphical representation of ratios (Schliemann, Goodrow, & Lara-Roth, 2001) shows that third-graders can deal with the representation of points in a graph and that they can start understanding how straight lines in a graph represent the same ratio. These ideas do not arise spontaneously but seem to require situations carefully structured to draw upon students' former knowledge while introducing new mathematical representations. In our classroom studies we had to build these situations and engage children in discussions where different approaches are proposed and considered by the instructor and by the other children. In this sense, the discussion was much closer to an everyday problem-solving situation than to a traditional mathematics classroom focused on the transmission and application of rules. But, differently from what usually occurs outside of schools, here the children had access to the new representations we introduced.

We argued elsewhere (Brizuela, Schliemann, & Carraher, 2000) that young children's notations constitute tools to further understanding and thinking processes. Here we saw how algebraic notation and graphical representation help children consider mathematical relationships that are otherwise unwieldy if not intractable.

We look forward to seeing theories of mathematics education that draw upon findings from developmental psychology and everyday mathematics. We also look forward to seeing the cognitive developmental theories benefit from careful analyses of curriculum issues and of mathematical learning in schools. If human development is not only an individual but also a cultural enterprise, we need theories that show how culture and thinking come together.

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