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ORIGINAL ARTICLE

Early algebra and mathematical generalization

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Abstract We examine issues that arise in students' making of generalizations about geometrical figures as they are introduced to linear functions. We focus on the concepts of patterns, function, and generalization in mathematics education in examining how 15 third grade students (9 years old) come to produce and represent generalizations during the implementation of two lessons from a longitudinal study of early algebra. Many students scan output values of $f(n)$ as n increases, conceptualizing the function as a recursive sequence. If this instructional route is pursued, educators need to recognize how students' conceptualizations of functions depart from the closed form expressions ultimately aimed for. Even more fundamentally, it is important to nurture a transition from empirical generalizations, based on conjectures regarding cases at hand, to theoretical generalizations that follow from operations on explicit statements about mathematical relations.

1 Introduction

We examine students' generalizations about certain linear functions involving seating arrangements. The setting is two lessons in a third grade classroom in which the instructor directs students toward conventional representations (closed form expressions of mathematical functions). Our goal is to understand the role empirical generalization and conjecture play in their thinking. An underlying question is whether such reasoning can play a useful role in the trajectory towards mathematical generalization proper and algebraic reasoning or whether it is antithetical to long-term goals.

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1.1 Background

Mathematical generalization involves a claim that some property or technique holds for a large set of mathematical objects or conditions. The *scope* of the claim is always larger than the set of individually verified cases; typically, it involves an infinite number of cases (e.g., "for all integers"). To understand how an assertion can be made about "all x" we need to consider the grounds on which the generalization is made.

The grounds for generalization differ considerably in mathematics and early mathematics education. In mathematics, it matters not how a person understood the issues, how insight came about, how learning progressed. The focus is on the mathematical content and validity of the claim rather than the psychological world of the learner. A generalization (often referred to as a theorem) is taken to be true if and only if it is supported by a valid proof. On the other hand, in mathematics education, particularly early mathematics education, we cannot ignore the psychological world of the learner. We take a broader view regarding both the forms of reasoning and the grounds for assertions. We consider not only how students make use of introduced, conventional notation and techniques but also how they represent and reason about mathematics in their own ways. We seek to determine the grounds for their claims, recognizing that what compels them to draw conclusions and make generalizations may not fully conform to the accepted norms of mathematics. The aim is not to replace the standard fare of mathematics with the ideas, representations, and reasoning evinced by students. We need to understand and promote the transition from a mathematics grounded largely in empirical observation and particular cases to one based on logical coherence and, ultimately, reasoning about mathematical structures that have little or no footing in the empirical world.

Recent changes in educational policy have provided special reasons for examining how young students learn to make mathematical generalizations (National Council of Teachers of Mathematics, 1989, 2000). Noteworthy among these is the recommendation that algebra play an important role from the time students begin to study mathematics. NCTM's endorsement of algebra for young learners reflects a change in thinking among a significant part of the mathematics education community about what young students are capable of and what should be going on in elementary school classrooms.

NCTM's characterization of algebra for K-12 students is instructive:

> Algebra encompasses the relationships among quantities, the use of symbols, the modeling of phenomena, and the mathematical study of change. The word algebra is not commonly heard in elementary school classrooms, but the mathematical investigations and conversations of students in these grades frequently include elements of algebraic reasoning. These experiences present rich contexts for advancing mathematical understanding and are an important precursor to the more formalized study of algebra in the middle and secondary grades (National Council of Teachers of Mathematics, 2000, p. 37).

The view of algebra proposed by NCTM differs in significant ways from that encountered in secondary school and beyond. In particular, there is a heavy emphasis on students' learning to make generalizations about patterns. This is no doubt based on the fact that young children do not know what algebraic statements mean and hence must establish their initial meaning on the basis of situations and activities of an extra-mathematical nature.

We say extra-mathematical for a good reason: a *pattern* is not an acknowledged, much less well-defined, concept in mathematics. Textbook publishers, teachers, and students take wide ranging and inconsistent approaches to patterns, their properties, and their operations. Not surprisingly, many mathematics educators have found that it may be very challenging to get students from patterns to algebra (Schliemann, Carraher, & Brizuela, 2001; Lee, 1996; Mason, 1996; Moss, Beatty, McNab, & Eisenband, 2006; Orton, 1999). Students may predict the next element (or state) in an ordered set, yet find it difficult if not impossible to generalize, to generate a rule for determining the value of an element at an arbitrary position (Hargreaves, Threlfall, Frobisher, & Shorrockes-Taylor, 1999; Orton & Orton, 1999; Bourke & Stacey, 1988; Stacey, 1989). Extending an ordered set of objects shows some degree of generalization. But this falls short of an explicit generalization expressed in language or conventional mathematical forms.

Not all of students' difficulties in generalizing about patterns are of their own doing. They are often asked to reason about cases for which more than one rule might be reasonably inferred. Consider the ordered set, 1, 2, 4. Assume that it is extendible ad infinitum. Is the next element 6? Or 8? How about any integer greater than 4? Any integer not less than 4? Any integer at all? Any real number? Anything at all?

A pattern is not a mathematical object. Even mathematicians who claim that mathematics is the science of patterns would admit they are using the term in an extra-mathematical, almost poetic, sense. There is no agreement among mathematicians about what patterns are, nor about their properties and operations. This is a definite drawback if one hopes to move students towards mathematical generalization based on rigorous inference.

Nonetheless, generalization is not merely about rigorous inference. There is an important role for conjecture in mathematical generalization. In our view, the concept of *function*, which has a fairly long and established history, can provide trustworthy footing for the task at hand—much more so than patterns can. Functions can also be introduced in situations where students are encouraged to make conjectures.

Functions are normally introduced through algebraic expressions, but this option is not viable for young students who are unfamiliar with algebraic notation. This makes it imperative that we pay special attention to how functions are represented to students and by students. How do they learn to describe functional relationships and gradually express them according to the conventions of algebra? This raises issues dating back to Plato's Meno dialogue. How can one understand something that relies on advanced ideas or conceptual structures that one does not already have (Bereiter, 1985)? How can students make mathematical generalizations if they do not already possess the requisite understanding (Duckworth, 1979)? From what does such understanding arise?

To address these questions, we will first briefly describe the results of recent early algebra studies in the USA and Canada and elaborate on the concepts of function and generalization. We will then examine how third grade students participating in a longitudinal study on early algebra come to produce and represent generalizations during the implementation of two lessons.

From a mathematical perspective, the problems we will be considering are about the linear functions $g(t) = 4t$ and $f(t) = 2t + 2$. The problems are introduced through a narrative about an imagined social gathering at which guests are to be seated at dinner tables. We will deal with questions such as: How do students explain the underlying "seating capacity" functions? What varieties of generalization do students produce? How, if at all, does a narrative about a social event become a general statement about the functional dependency of one variable on another? What challenges do students face in their efforts to express seating capacity for an arbitrary number of tables? What sorts of intermediary representations serve to help students shift from a particular event, described in words, to a generalization expressed succinctly according to conventions of mathematics? What roles might teachers play in helping students learning to make mathematical generalizations of this sort?

1.2 Generalization in early algebra studies

1.2.1 Grounding in quantities and their relations

In recent decades the view that algebra should be taught in elementary school has gained prominence in mathematics education research, practice, and policy (Davis, 1967a, b, 1985, 1989; Davydov, 1991; Kaput, 1995; Kaput, Carraher, & Blanton, 2007; Kaput, 1998, b; National Council of Teachers of Mathematics, 1989, 2000; RAND Mathematics Study Panel, 2003; Schoenfeld, 1995). This so-called *early algebra* is not quite the same algebra that many readers recall from their eighth or ninth year of schooling. *Early algebra* interweaves with traditional topics of the elementary school curriculum, introduces algebraic notation gradually, and relies heavily on rich background contexts (Carraher, Schliemann, & Schwartz, 2007).

Such adjustments in mathematical content and approach are meant to address how young students (and perhaps older students as well) learn. Young learners acquire and enrich their understanding of mathematics by reflecting on situations involving physical quantities as well as numbers (Fridman, 1991; Inhelder & Piaget, 1958; Piaget, 1952; Schwartz, 1996; Smith & Thompson, 2007). This does not mean that mathematical knowledge is of a purely empirical nature, as Mill (1965/1843) unconvincingly argued. But with young learners it would be equally mistaken to suppose that mathematical generalization is a purely deductive enterprise. Mathematical objects cannot be displayed directly; they need to be embodied in some representational form. For young learners, the standard form of representation, algebraic notation, is initially not an option. Generalizations need to arise in activities associated with rich experiential situations. Many mathematical generalizations young students learn to produce stem from thinking about how physical quantities change or remain invariant as a result of actions and operations. Accordingly, one expresses the values of physical quantities through what are often referred to as *concrete numbers* (Freudenthal, 1973) of which two sorts are sometimes distinguished: *counts*, such as 41 blackbirds, and *measures* (scalar additive) such as 3.4 miles (Fridman, op. cit.).

1.2.2 Arithmetic as inherently algebraic

A common thread of research about algebra in elementary school in the USA and Canada is the belief that a deep understanding of arithmetic requires mathematical generalizations and understanding of basic algebraic principles. Let's briefly look at the main findings of these studies (for detailed reviews on these, see Carraher & Schliemann, 2007; Rivera, 2006).

Bastable and Schifter (2007) and Schifter (1999) describe examples of implicit algebraic reasoning and generalizations among elementary school children when discussions and reasoning about mathematical relations is the focus of instruction. Carpenter, Franke, and Levi (2003), Carpenter and Franke (2001) and Carpenter and Levi (2000) show fairly young children talking meaningfully about the truth or falsity of issues such as "Is it true that $a + b - b = a$, for any numbers a and b?" and producing generalizations about arithmetical principles. Blanton and Kaput (2000) describe third graders' robust generalizations and supporting arguments for general statements on operations on even and odd numbers, considering them as placeholders or variables. In The *Measure Up* Project, a curriculum inspired by Davydov's (1991) ideas on measurement and representation of quantitative relations (Dougherty, 2007), first grade children learned to produce generalizations about mathematical relations as they use part-whole diagrams or line segments to explain, for example, that, given two magnitudes A and B and the relationship $A > B$, $A - B$ is itself a magnitude that can be added to the lesser quantity (B) or subtracted from the greater one (A) to make the two quantities equal.

Issues specifically related to patterns and functions were explored by Moss et al. (2006) among second and fourth graders learning about the functional rules governing visual/geometric and numerical patterns as they participated in experimental curricula. They found that second and fourth graders can generate geometric patterns based on algebraic representations and find functional rules for patterns, moving towards understanding the *relationships* *between the two variables* in the problem. However, students were not taught to use letters to represent variables and the algebraic expressions for functions, a focus in the present study.

Our approach to early algebra and to mathematical generalization relies heavily on the concept of *function* (Schwartz, 1999; Schwartz & Yerushalmy, 1992a, b, c). We have proposed that arithmetical operations themselves be conceived as functions (Carraher, Schliemann, & Brizuela, 2000, 2005; Schliemann, Carraher, & Brizuela, 2007). Even though Usiskin (1988) contrasts algebra as generalized arithmetic and algebra as the study of relationships among quantities, we have bridged both approaches by focusing on algebra as a generalized arithmetic of numbers and quantities and as a move from computations on particular numbers and measures toward thinking about relations among sets of numbers, thus treating arithmetic operations as functions.

Our lessons often involved linear functions, and a few dealt with non-linear functions. Central to our approach is the use of problem contexts to situate and deepen the learning of mathematics and generalizations and the use of multiple representations, namely, natural language, line segments, function tables, Cartesian graphs, and algebraic notation. One has to be aware, however, that while contextualized problems and focus on quantities help in providing meaning for mathematical relations and structures, algebraic knowledge cannot be fully grounded in thinking about quantities (see Carraher & Schliemann, 2002a; Schliemann & Carraher, 2002). Moreover, as we have shown before (Schliemann, Carraher, & Brizuela, 2007), young students benefit from opportunities to begin with their own intuitive representations and gradually adopt conventional representations, including the use of letters to represent variables, as tools for representing and for understanding mathematical relations.

Our three longitudinal classroom studies, implemented over the last 10 years in Public schools in the greater Boston area, have shown that 8–11 year-olds can learn to (a) represent and grasp the meaning of variables; (b) shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures; (c) shift from computing numerical answers to describing and representing relations among variables; (d) build and interpret graphs of linear and non-linear functions; (e) solve algebraic problems using multiple representation systems such as tables, graphs, and written equations; (f) solve equations with variables on both sides of the equality; and (g) inter-relate different systems of representations for functions (see Brizuela & Schliemann, 2004; Carraher & Schliemann, 2007; Carraher, Schliemann, & Schwartz, 2007; Schliemann, Carraher, & Brizuela, 2007; Schliemann et al. 2003).

Before we describe data on children's generalizations in our early algebra classrooms, we will broadly define the concept of function and then mention a subclass of particular interest for our present purposes.

1.3 Functions, numbers, and quantities

1.3.1 What are functions and how are they related to generalization?

Most modern definitions of function are consistent with the Dirichlet-Bourbaki notion:

> A *function* is a relation that uniquely associates members of one set with members of another set. More formally, a function from A to B is an object f such that every $a \in A$ is uniquely associated with an object $b \in B$. A function is therefore a many-to-one (or sometimes one-to-one) relation. (Weisstein, 1999, viz., Function).

Strangely, this definition is so abstract as not to require that generalization be involved. It includes, for instance, functions that cannot be expressed by an algebraic formula such as a mapping of people's names to their shoe size (assuming each person has one and only one shoe size).

Young learners are commonly introduced to a narrower yet, for them, more suitable, view of function, according to which there exists a *rule of correspondence* for generating a single value in set B from each element in set A (Vinner & Dreyfus, 1989). Of paramount interest are those cases that can be described by an algebraic expression. We will refer to such functions as *formula-based*. The values of formula-based functions can be determined without a lookup table or extra-mathematical resources (e.g., a database, yearbook, report, the process of measurement, etc.). Formula-based functions need not be continuous. For example, the seating capacity functions we will be considering are not continuously defined, as they are, over whole numbers.

Formula-based functions are a special, "generalizable" subset of functions: the formula or algebraic expression (in this paper we use the expressions *formula*, *algebraic expression*, and *algebraic formula somewhat* interchangeably) is a means for determining the value of the function for *any* input (that is, any element of the domain). An expression of the rule of correspondence—whether formulated in spoken language, algebraic notation, graphs, diagrams, or some combination of such symbolic representations—constitutes a generalization about the relation between values in the domain and co-domain.

1.3.2 The importance of functions in school algebra

In school settings algebra is commonly introduced *without emphasizing functions*. The focus falls instead on "solving for x" in equations, where x is treated as a single unknown number.

The functional perspective broadens the meaning of algebraic expressions by treating "x" as a variable, that is, as an object that may vary in its value. Students are encouraged to move from thinking about operations on specific numbers to relations among variables.

A functions-based approach to equations is very different from solving for x. Students are encouraged to recognize the multitude of values each of two functions can attain before shifting their attention to the special conditions under which the functions are constrained to be equal. Setting two functions as equal does not signify that they are interchangeable (Carraher & Schliemann, 2007). In an equation such as in $3x = 2x + 7$ the equals sign introduces constraints not inherent to the definition of either function; that is, the functions $3x$ and $2x + 7$ are not one and the same function. It is important that students understand how this use of the equals sign differs from that employed in arithmetical expressions such as $5 + 3 = 8$, where the expressions on each side are indeed interchangeable, referring to one and the same number.

In the early going, students learn to make generalizations in situations involving physical quantities. They learn to use tables, graphs, algebraic notation, and other mathematical representations to capture general aspects of their reasoning about such situations. Gradually they become comfortable using letters to stand for variable quantities and operating directly on algebraic expressions. Only at fairly advanced stages do students reason at long stretches within the syntactical constraints of these symbolic systems.

1.4 Two expressions of a formula-based function

In mathematics there are two basic ways of using algebra to express a linear function: through a closed form expression or through a recursive formula. The more common, *closed form*[1] expression (Cuoco, 1990) consists of a statement such as that in Expression 1.

$$f(x) = 3x + 7 \quad \text{where } x \in \mathbb{N}_0. \tag{1}$$

[1] The hypergeometric sense of *closed form* (Weisstein, 1999, viz., Closed form, Generalized hypergeometric function) is not related to the present usage.

Recursive[2] *or iterative* expressions of a function, far less common in K-12 mathematics, consist of two expressions—one for an initial condition and another for all other cases (see Expression 2):

$$f(0) = 7 \quad \text{and} \quad f(n) = f(n-1) + 3 \quad \text{where } x \in \mathbb{N}_0. \tag{2}$$

Expressions 1 and 2 define the same function. However they reflect different *conceptualizations* of the function. The constant of proportionality, 3, appears in Expression 1 as the familiar multiplier, the a in $ax + b$. In Expression 2 that constant of proportionality appears as an increment in the repeating condition. Note also that the closed form expression allows one to compute the value of $f(n)$ in a single step. The iterative variant requires that we start at $f(0)$ and build our way up until we have reached the desired $f(n)$. This is tedious when n is a large number.

Were mathematics education to end with linear functions over the whole numbers, the curriculum might give equal time to each kind of expression. However closed form expressions are more efficient, requiring a single computation whereas recursion requires computations for $f(0)$ through $f(n)$. Additionally, closed form expressions can be easily extended for functions in the domain of real numbers. Much of the drama of early mathematics education unfolds against students' evolving concept of number, including rational numbers (fractions, decimal fractions, etc.), integers, and real numbers. The closed form approach continues to serve for each of these cases. The iterative approach does not. [The skeptical reader is challenged to use Expression 2 to find the value of $f(\pi)$.]

Despite the advantages of close-form expressions, we cannot ignore the recursive approach. It will later prove useful to the topic of differential equations. But more importantly, for the present discussion, the recursive expression tends to be consistent with many young students' conceptualizations of linear functions.

2 Expressing seating capacity as a function of the number of dinner tables

Here we report on two lessons regarding the representation of linear functions in a classroom attended by 15 students who took part in 3-year longitudinal investigation in two classrooms from a school in metropolitan Boston. The children, who lived in the local minority and immigrant community, participated in the investigation's early algebra lessons since the beginning of third grade for 3 hours each week, in addition to their regular mathematics classes.

[2] Recursion and iteration refer to a process that "runs again and again." We are not using *recursive* in the logic programming sense of recursion, according to which algorithm execution begins with the repeating condition.

The lessons were videotaped and the students' written work was collected and analyzed.

2.1 The tasks

The two successive lessons analyzed here were implemented during the latter half of grade three, when the students were 8 and 9 years of age. The previous 33 lessons aimed at helping students to note and articulate the general relations they see among variables. For instance, in one of the first lessons (see Carraher, Schliemann, & Schwartz, 2007, for a description of this lesson as implemented with another group of students), the instructor holds a box of candies in each hand (John's and Mary's boxes) and tells the students that each box has exactly the same number of candies inside; all of John's candies are in his box and Mary has three additional candies resting atop her box. Children are then asked to write or to draw something to compare John's and Mary's amounts. As expected, most students include in their representations an exact amount of candies that would be in each candy box and add to that value the three extra candies for Mary. We then suggest, sometimes building on students' statements that they can't know for sure how much is in each box or on their use of question marks for possible amounts in each box, that we use a letter to represent any number of candies that could be in a box. After some discussion, third graders are able to adopt, for instance, N to represent John's amount and $N + 3$ to represent Mary's amount. The Candy Boxes lesson was followed by lessons where children explored relationships between variables across a variety of representations such as number lines, function tables, and algebra notation.

The lessons of interest involved the problem of seating guests at a dinner party. Students were first told the conventions for placing seats around dinner tables (one and only one guest along each table edge, no seating at corners, etc.). They were then asked to draw various numbers of tables and to keep track of how many people can be seated around them. Eventually they were asked to state and write a general rule for determining the number of seats based on the number of tables. Neither the rule of correspondence nor the rule for adding additional tables was explicitly stated, but each was implicit in the instructions about how to seat guests. It was the students' task to give the function's rule of correspondence in clear expression.

Two versions of the problem are discussed. In the Separated Dinner Tables version, four people can be seated at each table. In the Joined Dinner Tables version, tables are set end to end in a line that grows with each additional table. The reader will recognize the Separated and Joined Tasks as corresponding to the functions in Expressions 3 and 4, where n refers to the number of tables and $f(n)$, $g(n)$ refer to the number of people that can be seated at n tables, respectively. The tasks were introduced to the third grade students in the form of a word problem. Students were told how dinner tables are to be arranged, how seats are placed around the tables, and that each seat would be occupied by one person. However, the problems were not presented to the students in algebraic format such as:

$$f(n) = 4n \quad \text{where } n \in \mathbb{N}. \tag{3}$$

The Separated Dinner Tables lesson served as a warm-up exercise for the more difficult case of joined tables, treated in the following lesson. We chose not to emphasize the notions of domain and co-domain during these lessons. It was implicitly understood that we were using the set of whole numbers as domain and co-domain. The underlying function for joined tables is represented by Expression 4:

$$g(n) = 2n + 2 \quad \text{where } n \in \mathbb{N}. \tag{4}$$

2.2 The lesson about separated dinner tables

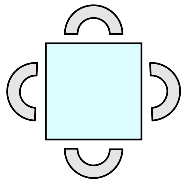
In this first lesson the instructor, Bárbara (who was also a member of the research team) presented the problem to the children and described the conditions for seating additional guests. It was important for students to accept that each table was to be filled; otherwise there would not be a unique output value for a input value—a necessary condition for any function. The constraint of filling each dinner table was clarified during a 10-min discussion about the terms *maximum* and *minimum*. *For a separated table*, "zero seats" was legitimate minimum; one, two, and three seats were possible values; a maximum number was 4. The students task was to determine the maximum number of people that could be seated at varying numbers of dinner tables when they were joined together.

Bárbara then distributed the lesson handout for the students to complete (Fig. 1). The sequence was for 1, 2, 3, and 9 (joined) dinner tables. A tabular representation with regular intervals between input values is known to encourage a recursive strategy or scalar approach (Vergnaud, 1983). The gap between 3 and 9 tables was intended to interrupt such a column-wise, building-up strategy and to encourage students to consider within-row, input–output, an approach considered more congenial to closed form expressions of the function (see Schliemann, Carraher, & Brizuela, 2001).

After the students completed the handout, Bárbara reviewed with them the seating capacity for each case, registering the agreed-upon results in tabular form on the whiteboard. One student noted that when the rows of the data table are not arranged in a regularly increasing fashion, it is difficult to make use of the values in one row to obtain to values in the next. This is precisely the sort of

Fig. 1 Handout for the first lesson

Handout: Detached Tables



Name: ______________________________

In your restaurant, a maximum of four people can sit at each dinner table.

Fill in the following data table.

If you know the number of tables, figure out the maximum number of people you can seat.

If you already know the number of people, figure out the minimum number of tables you need.

Number of Dinner Tables	Show How	Number of People
1	1 × 4 →	
2	2 × 4 →	
3	→	
4	→	
	←	24
	←	20
	←	11

How many people can you seat at **t** tables? [hint: More than **t** people? Less than t people? Exactly **t** people?]

How many tables do you need to seat **n people**? [hint: More than **n** tables? Less than **n** tables? Exactly **n** tables?]

interruption mentioned above. But, as we shall see, this measure in no way ensures that students will re-conceptualize the problem in terms of a mapping of inputs to outputs.

Children proceeded to complete their handouts individually or in small groups. They filled in the data table in the handouts using various methods: by multiplying the number of dinner tables by 4, by adding 4 plus 4, plus 4 repeatedly (once for each dinner table), and by adding 4 to the number of people in the prior row to obtain the number of people in the next row.

The ensuing discussion focused on the meaning of the written computations within the dinner table context. Children at first described what we might term 'number of people per table' as 'number of people'. Bárbara systematically rephrased children's statements or guided them to do so, insisting on the differentiation between total number of people and number of people per table. A student (Aaron), building on previous work on notation for variables, proposed to use p for the number of tables and stated that '4 times p' expresses the total number of people.

At one point in the discussion, issues regarding the inverse function, $n \rightarrow n/4$, arose. To answer the question "What if we have 20 people total?" Brianna said that 4×5 is 20 and that you can think of 20 divided by 5 and the answer is 4. Bobby proposed that one can also think of 20 divided by 4. Marisa viewed the question as: 4 times what number equals 20? Bárbara asked the children to consider the operation they were using when they were given the input to get the output and vice-versa. During the discussion about inverse functions (inferring the number of tables from the number of seats) we realized that, in the last row of the data table in the handout (see Fig. 1) we had mistakenly introduced a question that turned out to have no correct answer: The value, 11 people, does not belong in the range of the function and does not correspond to the maximum number of people that can sit at three tables nor to the maximum of four people). Perhaps not surprisingly, students proposed to have two tables with four people and one table with three people. But this case leads to an inconsistency about the domain (and hence about the very function we are talking about). After the class, we realized it would be in the interest of consistency to list only maximum seating in future enactments of the lesson.

At the end of the class, children were given for homework a similar problem involving triangular dinner tables. Later the children would work with hexagonal tables. These variations were intended to serve several aims. They let students have extended practice on similar problems. In addition they introduced the notion of a family of functions having the same structure (they were all functions of direct proportion) but varying in the value of a parameter, the coefficient of proportionality.

2.3 Reasoning about the seating capacity of joined dinner tables

In the second lesson the square dinner tables were to be arranged end to end (see Fig. 2).

Bárbara began by reviewing the previous lesson. Students recalled that they were determining the maximum number of people that could sit at the tables in a restaurant and that the maximum number of people per table was 4. Anthony recalled that one has to "times the number by four to get the answer" and concluded that, for 11 tables, you can seat a maximum of 44 people. Bárbara asked how many people would sit at 100 and at 1,000 tables, and Deshawn and Aja answered 400 and 4,000, respectively.

Five minutes into the lesson Bárbara introduced the new rule for the arrangement of tables:

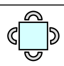
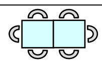
Bárbara: … when a lot of people show up together, as a party, they start putting tables together, this way […]. If that is a square table like this [pointing to a square] we can have four people… How many people can sit at two tables that are together? Eh, Deshawn…
Deshawn: 8?
Bárbara: 8?
Aaron: No…
Bárbara: Well, before, when we didn't put the tables together there were 8, right Deshawn?
Deshawn: Equals 6 …
Bárbara: Well, let's see… (she counts along the outside edges of the tables in a drawing, arranged side by side) 1, 2, 3, 4, 5, 6… Now, how come that when we had two tables there were 8 and, now we have two tables and we have 6… Why Gio?
Gio: Can I show? (he walks to the overhead projector).
Bárbara: Why is it that, on Tuesday, when we have two tables we had 8 and now 6?
Gio: Because of these two sides, right here… (pointing, in the drawing, at the two inside edges where the tables meet and where guests can not sit.)
Bárbara: What is going on with those two sides?
Gio: I don't know how to say it!

Fig. 2 Handout for the second lesson

Handout: Attached Tables

Name: ______________________________

In your restaurant, square dinner tables have are always arranged together in a single line. Below, figure out the maximum number of people you can seat.

Dinner tables	Show how	Number of People
1 		4
2 		
3		
4		
5		
6		
7		

Each time you add another table to the line, how many new people can join the party?

[on the back side of this paper]: If I tell you the number of dinner tables lined up, how can you figure out the maximum number of people that can sit down?

2.4 Students' interpretations and attempts to generalize

2.4.1 Using words to express the underlying function ($n \rightarrow 2n + 2$)

Gio noted that when the dinner tables are joined, two seating places disappear but, as with other children, he cannot verbalize why. Anthony stated that if the tables are not joined there will be more people because "somebody can sit at the middle of the tables" probably meaning that these seats will be lost. Students agreed that two seats would be lost and discussed how many people would sit at two and at three joined tables. At this point a student attempted to generalize how seating capacity increased according to the number of tables: "You keep adding two…".

Bárbara then displays a chart (see Fig. 3) with headers for the number of dinner tables (t), the maximum number of people that could sit when the tables were separated (Tuesday column), and the maximum number of people that could sit when the tables were joined (Thursday column). Working together with the children, she fills in the results for two and three dinner tables, first when they are separate, and then when they are joined. As each row of data is completed she contrasts the results for the two situations. Two function tables are represented on the whiteboard side by side, one for each seating rule.

Anthony predicts that four joined tables will seat ten people. Bobby explains that it is because "you are adding two each time". Bárbara draws the dinner tables and, together with a few students, counts the sides, finding that

Fig. 3 Chart relating the number of dinner tables to number of people that could sit if the tables were separated (Tuesday column) or joined (Thursday)

ten people can be seated. Some students show surprise that it was ten. When Bárbara asks how many people can sit at five tables, Mehrose answers "12". Bárbara reviews the amounts in the table and a few students claim that two more people can join the party each time a new table is added.

Fifteen minutes into the lesson, Bárbara asks how many people can be seated at 100 tables. This crucial example (see Balacheff, 1987) was meant to prod students to abandon the iterative and time-consuming approach and look for another way to approach the problem. The children react with puzzlement. Marisa and other students predict that 200 people can sit at 100 tables. Gio proposes 101. At this point, it seems that students appear to be simply voicing guesses. They have not proposed a general rule or computation routine.

Bárbara reminds them of the previous lesson where the students had found a formula that works for any and all cases. She asks them to recall the rule to get from t (the number of tables) to p (the number of people), when the tables are separate. Several students mention that they had to multiply by four. After some discussion, with input from the children, Bárbara writes the equation $t \times 4 = p$ and asks the children to state what each of its elements refer to:

> Aja: The four stands for the "*4 people at the table*" [emphasis added here and below]
> Bárbara: *At each table*, right? Yes… number of *people per table*… at each table… What the '*t*' stands for? What does this '*t*' stands for, Hannah?
> Hannah: Tables!
> Bárbara: Number of tables, right? And, what does the '*p*' stands for?
> Aaron: People.
> Bárbara: People where?
> Student: *People that sit at that table*.
> Bárbara: The *total number of people*, right? *Because we already had the four people per table*. Now, this is the total number of people… Now, the hard thing is, now that you know the formula that worked on Tuesday, you need to figure out the formula for today (Thursday).

Bobby (pointing to each row in the data table on the board, shown in Fig. 3) considers how the seating capacity from joined tables differs from the seating capacity of the joined tables.

> Bobby: … right here (pointing to the number of people for two joined tables) you take away two (from the number of people for two separate tables), right here (pointing to the number of people for three joined tables) you take away four (from the number of people for three

separate tables), then you take away six (from the number of people for four separate tables), right here you take away eight (from the number of people for five separate tables), and right here you take away ten (from the number of people for six separate tables).

Bobby notices that, as the number of tables increases, joined tables hold increasing fewer guests than separate tables. Bárbara acknowledges Bobby's point, but wants Bobby to provide a general statement that does not rely upon the function for separate dinner tables.

Bárbara: Ok, each time you are taking away two more [from the potential increment of four seats]... Ok, that's a good thing you notice. But if there's any way that we could figure out… if we don't know… if we didn't have this column [for separate tables], if we cover it up, we need to figure out a way to get from 1 to 4, from 2 to 6, from 3 to 8… ok? So, I will give you a chance to think about that now, for a while… with your handouts… yes? I want you to figure out if there's any rule to get from this column (number of tables) to this column (total number of people that can sit at joined tables), ok? And we have a couple of predictions here. Does anyone else have a prediction? Anthony, do you have a prediction for 100 tables? Anthony…

At this point, Anthony correctly predicts that 202 people can be seated at 100 tables. Bárbara then proposes that the children work on their handouts in small groups.

2.4.2 Towards algebraic expressions

Most children completed the first part of the handout by drawing the dinner tables and counting the number of outside edges where a person might sit. Drawing the tables may appear unnecessary, given that they had already considered the same cases in a data table. Nonetheless, it is a means of convincing themselves that the data are correct by visually scanning the cases for change and invariance. By this point the students appear to understand that, each time one adds another table, two more people can be seated. This does not provide a direct answer to the question, "If I tell you the number of dinner tables lined up, how can you figure out the *maximum number of people that can sit down*?" However, as we will see next, the drawings constituted a path to the multiple types of generalizations the children came to develop as they were guided to find and represent a general rule corresponding to the closed form expression of the function.

While working in small groups, Bobby finds a general rule and describes it as if reading an algebraic notation aloud (see Fig. 4):

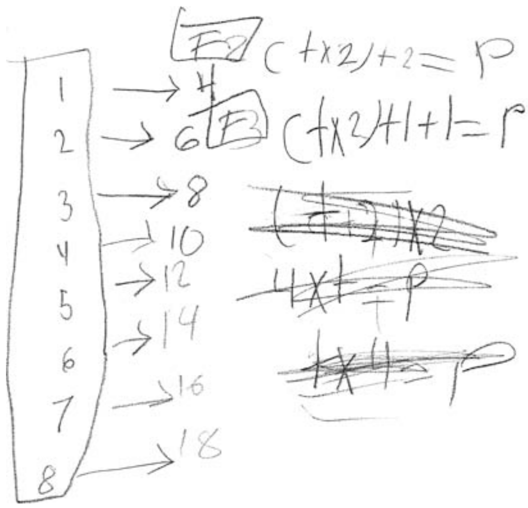


Fig. 4 Bobby's written work

Bobby: … *t times 2 plus 2 equals p*… [emphasis added]
Analúcia (repeating): *t* times 2 plus 2… This is very good. Now, would you explain it to me?
Bobby: Ok, it's like 2 times 2 equals 4, add 2. It's 4, then, 2 times 2 equals 4 plus 2 equals 6; and 3 times 2 equals 6, add 2, you got 8.

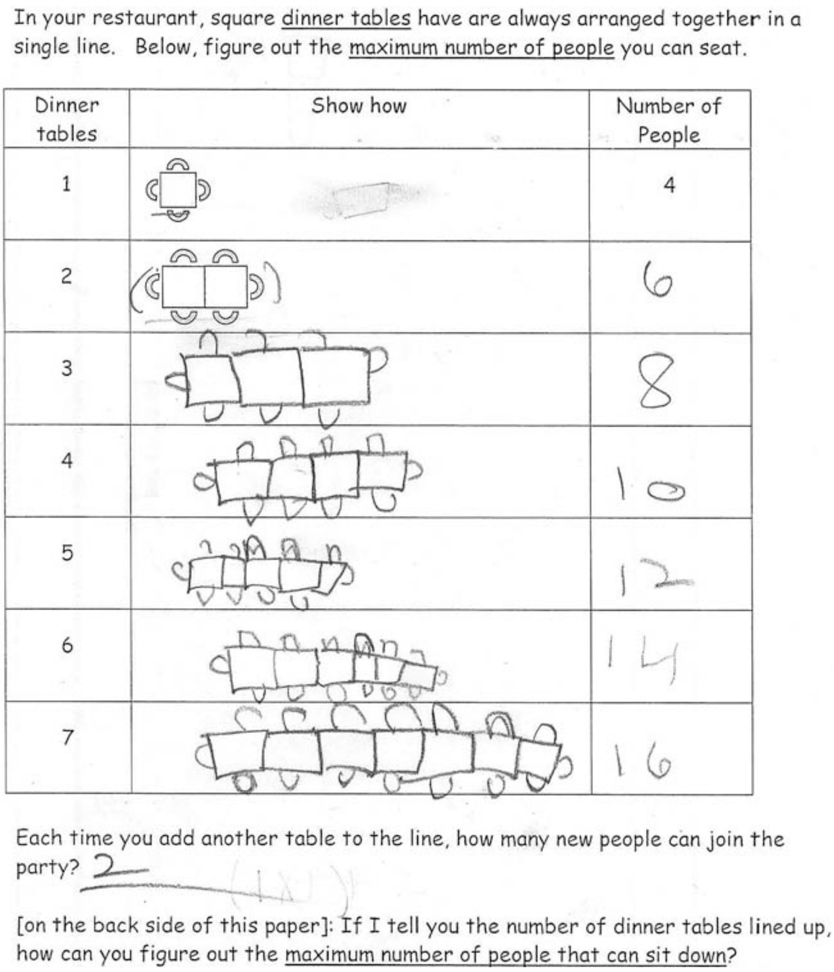
In another group, Gio struggled to find a general rule. He had generated drawings for each case (see Fig. 5) and successively added 2 for the next total. When Analúcia asked him to find out the number of people for 100 tables, he states that 202 people could sit at the tables and generates a general rule for the function.

Gio later explains the case of 100 tables to the whole class:

Gio: Coz if you have one table, on that side is 100, 100 over there [along the other long side] and then two over here [at the heads of the table]… 200 plus 2 equals 202.

Gio has parsed the maximum number of seats as a sum of a component directly proportional to the number of tables (2*t*) and the additive constant, 2, that corresponds to the ends of the line of tables. He easily computes the number of people that could be seated at 1,000 and at 200 tables but says that it is hard to show how to do this in a formula. Bárbara intervenes:

Bárbara: Gio was able to explain that, whatever number of tables you have, you have double sides, you have the side at the top and the side at the bottom and then you have two more people on the ends… you have to count the ends. So, can you try Gio? Can you try to make that up into a formula?
Gio: For 100?


In your restaurant, square dinner tables have are always arranged together in a single line. Below, figure out the maximum number of people you can seat.

Dinner tables	Show how	Number of People
1		4
2		6
3		8
4		10
5		12
6		14
7		16

Each time you add another table to the line, how many new people can join the party? 2

[on the back side of this paper]: If I tell you the number of dinner tables lined up, how can you figure out the maximum number of people that can sit down?

Fig. 5 Gio's initial work

Bárbara: For whatever number, no matter how many tables there are, what do you do if the only thing we know is the number of tables; t is everything we know. Everything we know is that we have t tables… what do we do with the number of tables?

Gio: Times 2 plus 2.

Bárbara: Times 2 plus 2, equals what? If $t = 100$, it will be 202 but what if we don't know? What would the answer be?

Gio: t times 2 plus 2.

Bárbara: Equals… the number of people, right?

Afterwards Mehrose suggests surrounding $t \times 2$ by parentheses. Bárbara writes the equation $(t \times 2) + 2 = p$ on the board and proceeds to discuss the meaning of each of its elements with the class.

The general statement $(t \times 2) + 2 = p$ constitutes a closed form representation of the function relating the number of dinner tables to the maximum number of people that could be seated. It was the target of the lesson, so to speak; namely, to have students coming up with equations for the functions $f(t) = t + t + 2$ or, ideally, $f(t) = 2t + 2$ (see generalization 3.2.1 below).

In the class handouts eight of the 15 children answered the question "If I tell you the number of dinner tables lined up, how can you figure out the maximum number of people that can sit down?" with a closed formula while the other seven relied on drawings and counting or didn't answer the question. Upon reviewing the videos of the lesson, we have found that, in their development towards the closed form representation favored by the instructor, they have produced representations that fall into the two overarching classes we have been discussing, that is, a recursive formula versus a closed form expression. As we will see, there were some variations within each of these two broad categories that prove to be enlightening.

3 Students' generalizations

Generalization refers to both the process and product of reasoning (viz., Radford, 1996). Different ways of "visualizing" a pattern are tantamount to different conceptualizations that may lend themselves to different algebraic expressions. As we shall see, a student who conceived of the table arrangement as a variation on the separated tables condition produced the expression $4t - 2(t - 1)$, which is somewhat different from a student who parsed the problem as the sum of seats along each of the four sides of the table configuration and expressed it as $t + t + 2$. However, both formulas represent the same function, in the same way as, in the history of mathematics, we find different ways of writing the "same equation" (see Filloy & Rojano, 1989).

Our analysis of children's generalizations takes into account the following dimensions of their conceptualizations: form of the underlying mathematical function (closed or recursive), variables mentioned (number of tables, auxiliary variable, number of people), types of operations used (addition, subtraction, multiplication), use (or not) of algebraic notation. We also consider the structural features (how many terms, use of parenthesis, etc.), the meaning of the different components of the written expression [i.e., considering the expression "$2t + 2$", the "+2" represents two people but if we consider the function $2(t + 1)$ the "+1" represents one table], on what elements the generalization is built on (numbers on the function-table, comparison of two function-tables, diagram of the dinner-tables, etc.), and referent transforming (i.e.: in the expression "$p(t) = 2t + 2$", the number "2" that multiplies t may work as a ratio that exchanges tables for people, two tables per one person).

We identified several types of generalization regarding the maximum number of people that can be seated at joined tables. These solutions are neither exclusive nor exhaustive, yet they appear to be ordered according to those that (1) treated the number of seats as a recursive sequence of numbers versus those that (2) treated the number of seats as a function of number of tables or as an input–output function explicitly linking two variables. Within category 1 we have found (a) recursive generalizations with "building up" in the output column, with a focus on the difference, $f(n) - f(n - 1)$ and (b) recursive generalizations with "building up" in an auxiliary column, with a focus on the difference, $f(n) - n$. Category 2 generalizations appeared in at least three different formats, focusing on: (a) Reckoning seats one table edge at a time, (b) Treating adjoined tables as a variation on separate tables with obstructed seats, and (c) inventing a virtual dinner table. In what follows, we describe students' generated rules that exemplify each type of generalization. The mathematical representations of these rules enlighten each rule's specific aspects.

3.1 Treating the number of seats as a recursive sequence of numbers

The focus of this family of approaches rests on the changes in the dependent variable (number of chairs) over time. The generalization is reached when one has formulated a statement that holds true for the initial condition and for each and every subsequent change.

3.1.1 *Focus on the difference, f(n) − f(n − 1)*

This approach typically arises when discussion takes place regarding data in a function table. The student draws attention to the increment by which the values in column two of one row can be used to find the value in column two of the next row. As Bobby expressed it, "… you are adding two [people] each time".

This increment has a direct counterpart in drawings of people seated around tables. Figure 6 (top) shows the case for an arbitrary number, t, of tables. This is the embodiment of information contained in row n. Figure 6 (bottom) represents the case where one more table has been added. In other words, it corresponds to the information contained in row $t + 1$ of a data table. The additional two people in Fig. 6 are shown as shaded. The person at the right end of the table is not a new arrival; presumably she merely "scoots over" each time a new table is added. The drawings in Fig. 6 are consistent with the iterative or recursive formulation of the function (see Expression 5)

$$\left.\begin{array}{r} p_1 = p(1) = 4 \\ p_t = p(t) = p_{t-1} + 2 \quad \text{where } t \in \mathbb{N} \end{array}\right\}. \qquad (5)$$

Students using this approach did not produce a formula expressing the functional relation. Also, they did not explicitly mention the starting condition. The approach is a *partial* recursive approach, whereby only the repeating condition is given expression. There is no referent transforming (Schwartz, 1996) here, as they do not use the input

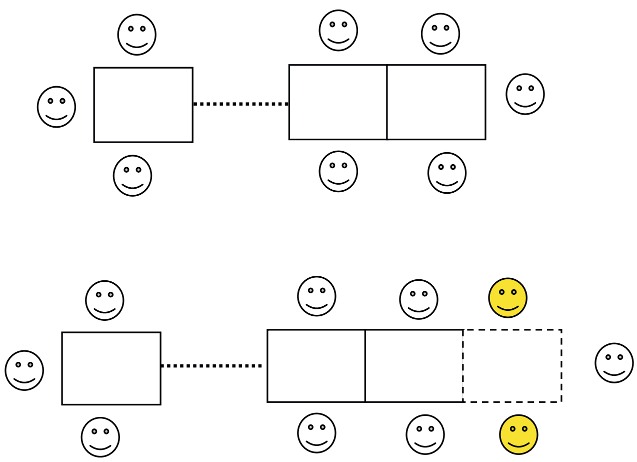


Fig. 6 Diagrams of t and $t + 1$ tables. Together they correspond to the repeating condition of a recursive expression, namely, $P_{t+1} = P_t + 2$ people

values as a starting point to produce the output values: they are already working with the number of people. To some extent, the iterative approach was encouraged by the fact that the problem was presented as a narrative that unfolds over time as more and more guests arrive. Students were also encouraged to add tables, one by one. Furthermore, the function table Bárbara had drawn on the blackboard was filled out in an ordered way (although some gaps or jumps were introduced).

This approach is based on the idea of underlying function as a recursive sequence. It can be considered as a *sequence*[3] because students treat the numbers in the output column as a set in which the order matters. Students identify recursion ("keep adding two") as the principle that generates successive values in the output column. From their perspective, the table in the handout sheet has missing rows. They commonly imagine the values of the "missing rows" in order to fill in the gaps.

In a sequence, the independent variable corresponds to the position in the ordered set. If one imagines the data table with all of its "missing rows" restored, then the position of each value of the function coincides with its row number. In the closed form variation, the independent variable tends to be construed as the *number of tables*. "Position in the sequence of values" and "number of tables" are somewhat different ways of thinking about the independent variable for seating capacity. But this distinction may be unimportant: when the values of dependent variable are displayed in ordered fashion, students tend to downplay or entirely ignore the independent variable, as if the sequence involved a single variable (number of chairs).

3.1.2 Focus on the difference, $f(n) - n$

This approach appears to arise in the context of data tables, where the values of the independent variable and dependent variable are listed as pure numbers, with no explicit units of measure. The student looks for a complementary number, such that, when added to the value in column one (independent variable) produces the known value in column 2 (the dependent variable). As described in a case study by Martinez and Brizuela (2006), by observing the pattern in both columns, the student constructs (mentally or actually) an auxiliary, intermediate column. The value in column 1 plus the value in the auxiliary column yields the value in column 2. This kind of building up fundamentally differs from recursion in the output column, discussed above.

[3] (Weisstein, 1999, viz., Sequence, Arithmetic progression) defines a sequence as an ordered set of mathematical objects, $\{o_1, o_2, o_3, \ldots o_n\}$.

Table 1 A data table showing the reasoning underlying the difference $f(n) - n$

Dinner tables, t	Increment to get number of people from the number of tables	Number of people (p) seated at t joined dinner tables
1	+3	4
2	+4	6
3	+5	8
4	+6	10
t	+"$t + 2$"	$2t + 2$
t	+"$(m - 1)t + b$"	$mt + b$

Hannah describes this type of generalization as follows:

Hannah: to get from 1 to 4 you have to add 3, then to get from 2 to 6 you have to add 4; for 8, 5; for 10, 6; ah…for 12, 7.

In the present case the numbers in this auxiliary column increase by ones (Table 1).

Interestingly, this approach appears to be hybrid of an input–output approach to the relation and a recursive approach. The auxiliary column only comes to be after the student has inspected the function table. After Hannah considers the function table on the whiteboard (see Fig. 3), she computes the difference between each output and its corresponding input, noting that the first difference is 3 and that each subsequent difference increases by one. She then proceeds to extend this sequence. In the two last rows of table 1, we provide our own algebraic representations of this way of generalizing. In the second to last row, we show the algebraic expression that captures the relation between the input and the numbers in the auxiliary column. In the last row, we provide an algebraic representation of this perspective when considering any linear function [not only the case for $f(t) = 2t + 2$]. However, students using this approach did not produce a formula describing the rule.

We want to emphasize that this generalization builds on the numbers displayed not only in the output column but also in the input column. In addition to that, a third column is created (mentally or in writing) to explain how the numbers in the output column can be created. This was not the case in generalization 1a, where students mostly ignored the input column and worked only with the numbers in the output column of the function table.

Because the model conceived by students was constructed purely by perusing the values of the function-table, students did not focus on coordinating the referents (tables, people) for the variables. The emergence of the model may have benefited from the fact that the semantics of the situation was suppressed or downplayed in the table of numbers. If we were to explicitly include the natural units associated with the numbers, the strategy would appear not to work. The number sentence associated with the first row in the function table, 1 + 3 = 4, is a true statement. However, if we add referents to the numbers we obtain: 1 [table] + 3 [people?] = 4 [people]. This is problematic since addition is not referent transforming in nature (Schwartz, 1996).

3.2 Treating the number of seats as a function of number of tables

A second family of approaches directs the initial attention not to change but rather to invariance. For example, a student may notice that, regardless of the number of tables, the number of seats around the joined tables will be equal to the sums of seats situated along the four sides of the composed tables. The student may then try to express the number of seats along each edge in some general way, that is, as a function of the number of constituent dinner tables.

3.2.1 Reckoning seats one table edge at a time

This was the case of Gio explaining how to determine the maximum number of people that can be seated at 100 tables:

> Gio: Coz if you have one table, *on that side is 100, 100 over there* [along the other long side] and then *two over here* [at the heads of the table]… 200 plus 2 equals 202. [In Fig. 7, we propose a diagram that illustrates Gio's generalization.]

Mathematically, this generalization corresponds to Expression 11:

$$p(t) = t + t + 2. \qquad (11)$$

For this case, students produced a closed formula representing the relation between the number of tables and

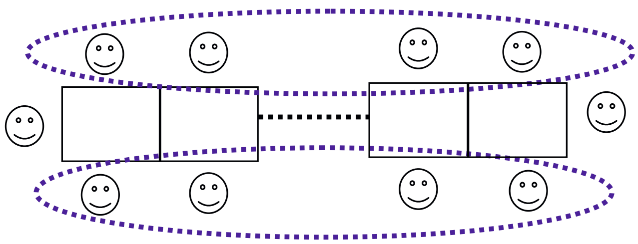


Fig. 7 A general way of counting the number of people seated at *t* joined dinner tables

the number of people that could be seated. During work in small groups, Gio used the recursive sequence approach until he was asked to solve the case for 100 tables. The question led Gio to look at his drawing of tables (Fig. 5) as a general diagram, so that it could stand for any number of tables from four upwards. This helped him to develop a strategy to determine the output. Differently from the former two strategies, this generalization builds on the semantics of the situation, not on relations among numbers.

Even though the algorithm works numerically, it is inconsistent with regard to the referents. For three tables, we might think of Gio's approach as follows: 3 [the number of people on one side] + 3 [the number of people on the other side] + 2 [the people at the ends] = 8 [people in all]. However, the given 3 is the number of tables. The algorithm, 3 [tables] + 3 [tables] + 2 [people] = 8 [people], is problematic. So, it seems that Gio was implicitly using a one to one correspondence between tables and people along a table side. Each table has two sides, so we can exchange each side for one person. In this way, even though the formula does not capture the referent transforming, Gio seems to have implicitly transformed the referent. Drawings such as Gio's (Fig. 5) contain explict information about the value of the input and output variable. And there is a one to one correspondence between the number of tables and the number of people seating along one side of the line of tables. Hence, Gio can conveniently undertake a referent transformation by moving from the number of tables to the number of people along one side. This exchange function is a subtle yet important feature that may be peculiar to this particular framing of the problem.

Later, Gio produced a different version of the formula, using multiplication. Like Bobby, who had generated the rule at an earlier point, he expresses it as '*t* times 2 plus 2'. Here Gio transformed the "$t + t$" into "$2t$". The referent transforming character of this approach can be captured in the following way: 2 [people/table] × *t* [tables] + 2 [people] = ($2t$ + 2) [people] = *p* [people].

3.2.2 Treating adjoined tables as a variation on separate tables: obstructed seats

This generalization builds on what is already known about the seating capacity for separated tables, namely that the number of people will be four times the number of tables. Viewed in this manner the value of the Joined Tables function will be equal to the value of the Separated Tables function minus the missing places. Bobby describes his solution as follows (see Table 2 for our rendering of his approach):

Table 2 Data table showing the reasoning underlying a generalization focused on the missing places

Dinner tables, t	People (p_s) at separate dinner tables, $p_s = 4t$	People (p_m) who will not sit between tables $p_m = 2(t - 1)$	People (p) at t joined tables $p(t) = 4t - 2(t - 1)$
1	4	0	4
2	8	2	6
3	12	4	8
4	16	6	10

The data table, including the headers, is the authors' rendering

> "all right… right here you take away 2 (total people on Tuesday − 2 = the total amount of people today for 2 [dinner] tables), right here you take away 4 (total people on Tuesday − 4 = total today for 3 tables), then you take away 6 (total people on Tuesday − 6 = total people today for 4 tables.) right here you take away 8, and right here you take away 10…"

Figure 8 shows a diagrammatic representation of this type of generalization.

Mathematically, we can represent each of the steps of this generalization as follows (Expressions 6–10):

$$p_s = 4t \quad (6)$$

$$p_m = 2(t - 1) \quad (7)$$

$$p(t) = p_s(t) - p_m(t) \quad (8)$$

$$p = 4t - 2(t - 1) \quad (9)$$

$$p = 2t + 2. \quad (10)$$

This type of generalization appears to have been elicited by the data table on the whiteboard showing the input (number of dinner tables), the output for separate tables, and the output for joined tables (Fig. 3). Students apparently calculated the number of people to be subtracted from the results from separated dinner tables to obtain the results for joined tables. No student produced a written expression for this generalization.

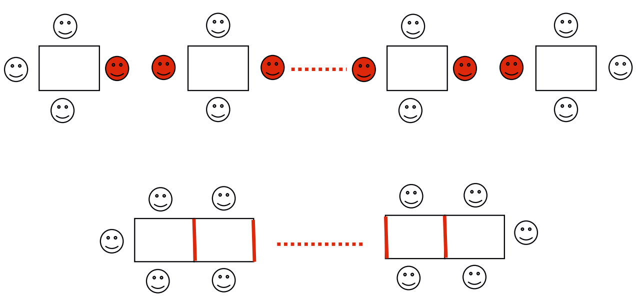


Fig. 8 Diagrams showing, *top*: the number of people seated at separate tables, corresponding to the algebraic formula $p_{st} = 4t$, and *bottom*: the final state after attaching the tables. The *shaded faces* indicate places unavailable after tables are joined

3.2.3 Inventing a virtual dinner table

Tarik proposes to add 1 to the number of tables and simply multiply by 2, saying: "Plus 1 and then times 2." Bárbara writes Tarik's rule as $t + 1 \times 2$ (thereby explicitly representing the independent variable) and, after some discussion about the meaning of each element, asks:

> B: Do you think there's something missing Tariq? [Looking toward another student…] Mehrose?
> Mehrose: Parentheses
> B: Parentheses… where should I put them?
> Melrose: in t plus 1…

Tarik's solution represents a re-conceptualization of seating capacity from $2t + 2$ to $2(t + 1)$. His procedure is not due to reflecting on the written form $(2t + 2)$ and factoring. It emerges from acting on the semantics of the situation rather than on the algebraic expressions. Tarik realizes that there are always two people at the ends to take into account and these two people have the same effect on the outcome as any intermediate table does. To clarify, one might imagine that an additional table has been added to the arrangement and that the two guests at the heads move to the sides of this new table (see Fig. 9). Once this additional, virtual table is introduced, one no longer has to take into account the heads of the table. The proposed formula not only satisfies the numeric relation but also can be made consistent with the referent transforming. For instance, 2 [people/table] $(t + 1)$[tables] = p [people]. This generalization is congenial to the following closed form algebraic expression: $p = 2(t + 1)$.

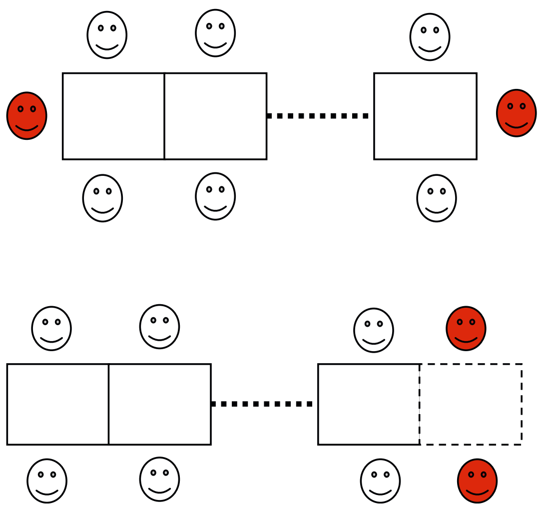


Fig. 9 Initially (*top*) we have t tables with two people per table in addition to the two people at the ends who are then placed at an extra table. This way we can count $t + 1$ tables with two people per table, obtaining the formula $p = 2\ (t + 1)$

4 Conclusion

There are two major issues in getting students to express mathematical generalizations about formula-based functions.

4.1 Issue 1: how do you get from recursive conceptualizations to closed form expressions of functions?

A recursive or iterative approach draws attention to changes as $f(n)$ becomes $f(n + 1)$. When the data are ordered the account takes the character of a narrative happening over time. If the independent variable always increases by one, it can be treated as indicator of position in the sequence or position in the narrative's time dimension. When one thinks in such a way, one treats the recursive function essentially as a recursive sequence. A recursive sequence may be congenial to students' thinking, but it is probably not where the mathematics teacher and textbook should be heading. And with good reason. When a function is viewed as nothing more than a sequence, students often fail to recognize the independent variable as a variable; it may remain implicit in their thinking.

We want students to use closed-form expressions of functions, but we often set them to thinking about functions as if we wanted them to provide us with recursive expressions. To some extent, data tables themselves, particularly when they are arranged so that the independent values increase by one, are at fault. But they permit a visual scanning of results that can be helpful for students to grasp how the function "works". The fact that input–output relations tend to be less salient than "building-up" relations does not mean that tables are necessarily impediments to understanding algebra. It does suggest that instructors need to pay attention to how students are parsing tables and representing the underlying functions.

It may be too much to hope that students will learn to express formula-based linear functions straightaway through closed expressions. So the issue becomes: how do we start off in one direction (towards recursion) and shift to another direction (towards input–output)? There are some activities, such as "Guess My Rule," that provide useful contexts for working with input–output expressions of linear functions (Carraher & Earnest, 2003). However, they are generally restricted to the domain of whole numbers (not quantities). Hence, the demands on modeling are substantially reduced. For example, no referent transforming is required for multiplication and division of integers.

In our lessons we employed two means to encourage this shift in conceptualizaton: (a) skipping numerical values in the independent variable [e.g., the case for 100 dinner tables] and (b) asking students to state the dependent value for an arbitrary but un-instantiated case (e.g., for *n* dinner tables).

As the students move towards generalization and closed form algebraic expressions they need to *explicitly represent* the independent and dependent variables in their statements and, in the case of linear functions, replace successive additions with a single multiplication. The latter step is challenging for students and teachers alike because it goes far beyond a substitution of computational operations. Multiplication is not simply repeated addition. It entails, implicitly or explicitly, transformation of referents (Schwartz, 1996) or an exchange function whereby the input units are changed into output units.

For students to determine the seating capacity for *any* number of dinner tables they need to consider much more than the last case. Their answer has to serve for all the data in the table as well as all data that would be generated by following the rules for seating additional guests. The closed expression captures the general relation between the numbers in the domain and the numbers in the image.

4.2 Issue 2: can empirical thinking lead to theoretical reasoning?

Generalizations are often distinguished according to whether they are empirical or theoretical. Empirical generalizations are thought to arise from an examination of the data for underlying trends and structure. Theoretical generalization is thought to spring from the ascription of models to data.

At first glance, these may appear to be variations on the same theme. However such generalizations have a dramatically different status in mathematics itself. A general, formula-based function over some infinite domain (e.g., counting numbers, real numbers) can not be defined through a finite set of ordered pairs (e.g., a table of data). It can only be defined through an explicit statement that captures its generality. To assume otherwise is to make a serious blunder from the perspective of mathematics.

But as we noted, functions cannot be introduced to young students through formal notation. The mathematical objects we want students to eventually be working with—variables, infinite sets of the domain and co-domain—are not, strictly speaking, *in* the problems we give to students. They must emerge from the activities and discussions about what, at first blush, are finite sets of mundane objects (tables, chairs, people) and their relations.

We can point to several moments where the teacher's work helped the students to construe the mundane problem in terms of relation between variables.

Let us recall when the teacher was interacting with Gio during the whole group discussion:

Bárbara: Yes, I know, it's hard, I know. But Gio was able to explain that, whatever number of tables you have you have double sides, you have the side at the top and the side at the bottom and then you have two more people on the ends… you have to count the ends… So, can you try Gio? Can you try to make that up into a formula?
Gio: For 100?
Bárbara: For whatever number, no matter how many tables there are, what do you do if the only thing we know is the number of tables, *t*, is everything we know. Everything we know is that we have *t* tables. What do we do with the number of tables?
Gio: Times 2 plus 2.

When Bárbara asks Gio to produce a formula, he seems lost. She rephrases the question giving him an opportunity to think. The way she restructures the questions takes into account the didactical history of the early algebra lessons, where the children had been introduced to letters, for example "*t or n*", to represent "*any number*" and produce formulas for the relationship between variables in different problems. The questioning seems to reach fruition when he produces a general rule.

The children developed different types of generalizations that were correct, but were not the ones we were expecting. The aim of discussing children's strategies when they deal with this kind of problem (or any other mathematical problem) is to stress the conceptual shift that some students need to go through to abandon their own models and construct something that is closer to the mathematical relationships and representations we want them to learn.

Gio initially approached the problem iteratively, adding two to the previous amount of people that could be seated in order to find out the total number of people for the next case. This strategy works if one wants to know the number of people that could be seated if one more table is added, knowing the number of people seated in $n - 1$ tables. But that was not the main question being asked. He was drawn towards the variation of people when the variation for the number of tables is one (Δp for $\Delta t = 1$). This is a correct relation for the problem and is also one of the questions in the task. But it does not lead to a general expression of the maximum number of people that can be seated at *n* tables. The teacher's question about the number of people that could seat at 100 tables finally seems to promote a shift in the way he approaches the problem: he can then produce the correct answer (202) without having to draw all the tables.

It might be argued that the children did not immediately arrive at a general formula because they were not paying enough attention or because they didn't understand what they were being asked to do. We think that this might have happen in some cases, but it does not explain what children actually do think when they try to answer what they are being asked. This kind of analysis allows us to intervene in terms of the mathematical content we want the children to focus upon. This was the case when Bárbara interacted with Bobby and told him that the focus should not be on the number of people seated at *n* separated tables, but rather on the *n* joined tables.

4.3 Mathematical notation and generalizations

Children's generalizations about the joined tables problem appear to have originated from acting on (mentally and physically) and reflecting upon their actions, deriving mathematical relations in the process. This appears consistent, in many ways, with Piaget's (1978) account of generalization. Children need to start from carefully crafted contexts and situations that may constitute physical analogues for mathematical structures. However, they need to go beyond the physical models and focus upon the logico-mathematical structures implicit in the models and eventually on the written notations themselves. In many ways this is similar to how professional mathematicians work, even though it may be strikingly different from how they think of their work, or represent it to the mathematical community in publications:

> "It is true that mathematicians also make constant use, to assist them in the discovery of their theorems and methods, of models and physical analogues, and they have recourse to various completely concrete examples. *These examples serve as the actual source of the theory and as a means of discovering its theorems, but no theorem definitely belongs to mathematics until it has been rigorously proved by a logical argument*. If a geometer, reporting a newly discovered theorem, were to demonstrate it by means of models and to confine himself to such a demonstration, no mathematician would admit that the theorem had been proved." (Aleksandrov, 1989, p. 2–3.)

Students also need to gradually appropriate the general representational tools of mathematics, which cannot be reduced to any particular experience or embodiment. In keeping with Davydov's (1990) and Dorfler's (1991) views, we consider the formalization in terms of mathematical notation an important feature in the mathematical learning process and in the development of generalizations:

> "When students meaningfully express ideas in the formal language(s) of mathematics, they transform their knowledge, extending its reach and meaning. Their formal representations can serve as objects of reflection and inquiry, thereby playing a role in the

future evolution of their mathematical understanding." (Carraher & Schliemann, 2002b, p. 299).

4.4 Quo Vadis?

Each of the issues raised above is related to a false dilemma. It appears unnecessary to force a choice between recursive and input–output approaches. Mathematics educators, theorists and practitioners alike, face the challenge of figuring out ways of moving back and forth between these representations. Students are typically more inclined to think of linear functions recursively (sometimes even as recursive sequences). But there are good reasons for introducing functions as input–output mappings.

We may wish to introduce generalizations to students through the very forms they are encountered in inside the field of mathematics. But, once again, this is inconsistent with how young students learn. They must first learn how to make mathematical generalizations about problems for which they are allowed to look for patterns and note relations and structures. Gradually they learn to formulate these generalizations using algebraic notation. Even more gradually, they will learn to derive new information by reflecting on the algebraic expressions they themselves and others have produced.

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